FCSD Semivariogram for Detecting Pure Tail-down Structure (Regular Rooted Binary Tree with q Points per Segment)

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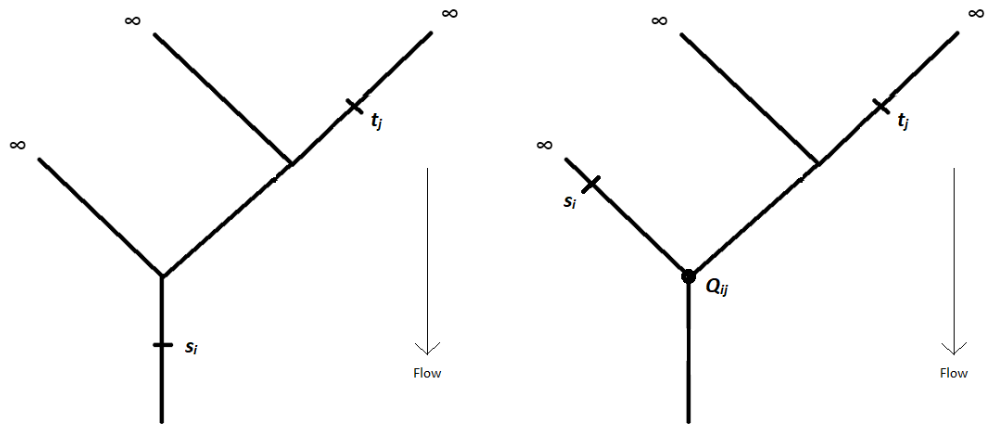
1 Introduction

1.1 Stream Network Topology

Stream network is a study region of geostatistics with very unique topological characteristics. It resembles $\mathbb{R}$ space in the sense that each edge (segment of stream) of stream can be characterized as line, but it also possesses $\mathbb{R}^2$ features since stream network branches at each node. Hence, special statistical tools and techniques should be used to handle the uniqueness of stream network.

Flow-connectedness and flow-unconnectedness

Assume that the stream network has no braided streams and no delta, in which case it has a single furthest downstream point, the "outlet," whose spatial coordinate is set to 0. Any location in the network can be connected to the outlet by a continuous curve along the network, and the length of that curve is called the "upstream distance" of that location. To uniquely define individual locations and keep track of their upstream distances, each location is denoted by s_i, where i indicates that the location is on the ith river segment, and s is its upstream distance (see Figure 1 below). Thus, two locations may be distinct but nevertheless have the same upstream distance. The "total stream distance" between locations s_i and t_j is the shortest distance between them through the network, and is denoted by $d(s_i, t_j)$. For each s_i, let U_i denote the set of stream segments that lie upstream of s_i, including the ith segment. Locations s_i and t_j are said to be "flow-unconnected" if $U_i \cap U_j = \emptyset$, and are "flow-connected" otherwise. If locations s_i and t_j are flow-connected, then $d(s_i; t_j) = |s - t|$. On the other hand, if s_i and t_j are flow-unconnected, then $d(s_i, t_j) = (s - q_{ij}) + (t - q_{ij})$, where q_{ij} is the upstream distance of the "common junction" of segments i and j, i.e., the junction where flows from segments i and j first combine.



(a) Stream distance $= t - s$ (b) Stream distance $= (s - q_{ij}) + (t - q_{ij})$

Figure 1: Flow-connected sites (left) and flow-unconnected sites (right) on a stream network

Tail-up and tail-down models

A classical approach to the development of covariance functions on the real line is to create model residuals as integrals of a moving-average function over a white-noise random process. i.e.,

$$\epsilon(s|\theta) = \int_{-\infty}^{\infty} g(x - s|\theta) dW(x),$$

where x and s are locations and $g(\cdot|\theta)$ is a square-integrable moving average function defined on the real line (Yaglom 1987). The covariance between between $\epsilon(s)$ and $\epsilon(s+h)$ so defined is given by

$$C(h|\theta) = \int_{-\infty}^{\infty} g(x|\theta) g(x - h|\theta) dx. \tag{1}$$

By the construction of $C(\cdot|\underline{\theta})$, it can be easy to verify that $C(\cdot|\underline{\theta})$ is a decreasing function of h.

Appropriate choices of the moving average function yield many of the covariance functions commonly used in Euclidean geostatistics (spherical, exponential, etc.). Cressie et al. (2006), Ver Hoef et al. (2006), and Ver Hoef and Peterson (2010) obtain covariance models for stream network variables by adapting this classical approach to the unique topology of stream networks. They consider in particular only unilateral models, i.e. those models that arise by taking the moving-average function to be positive in only one direction (either upstream or downstream) and zero elsewhere.

Considering those moving average functions that are positive only upstream, and weighting at each junction to achieve variance stationarity, one obtains the class of "tail-up" covariance functions

$$C_{tu}\left(s_i, t_j | \{\pi_{ij}\}, \underline{\theta}\right) = \begin{cases} \pi_{ij} C_{uw}\left(|s-t| | \underline{\theta}\right) & \text{if } s_i \text{ and } t_j \text{ are flow connected,} \\ 0 & \text{if } s_i \text{ and } t_j \text{ are flow unconnected,} \end{cases}$$

where the π_{ij}'s are spatial weights and $C_{uw}(\cdot)$, the unweighted flow-connected portion of a tail-up covariance function, is a valid covariance function on $\mathbb{R}$. Thus, variables at sites that are not connected by flow (e.g. sites on two branches upstream from their junction) are uncorrelated. The spatial weights are given by $\pi_{ij} = \prod_{k \in B_{ij}} \sqrt{\omega_k}$ where B_{ij} is the set of segments that lie between the ith and jth, including the jth but excluding the ith (the jth being downstream of the ith), and ω_k is any attribute of the kth segment (e.g., watershed area, flow volume, stream order, stream slope) expressed as a proportion. Often, it may be desirable to take ω_k to be the proportion of flow volume contributed by the kth segment to the junction at its downstream terminus, but flow volume is rarely available so watershed area, which is easily obtained via a GIS, may be used as a proxy. The lack of correlation between observations on flow-unconnected segments and the explicit account taken of flow volume (or a proxy for it) make tail-up models seem especially appropriate for stream network variables such as concentrations of point-source pollutants, which generally move passively downstream.

On the other hand, moving average functions that are positive only in the downstream direction yield the class of "tail-down" covariance functions

$$C_{td}\left(s_i, t_j | \underline{\theta}\right) = \begin{cases} C_{fc}\left(|s-t| | \underline{\theta}\right) & \text{if } s_i \text{ and } t_j \text{ are flow connected,} \\ C_{fu}\left(s - q_{ij}, t - q_{ij} | \underline{\theta}\right) & \text{if } s_i \text{ and } t_j \text{ are flow unconnected,} \end{cases}$$

where $C_{fc}(\cdot)$ and $C_{fu}(\cdot)$ are valid covariance functions on $\mathbb{R}$ and $\mathbb{R}^2$, respectively (and are related to each other through their functional dependence on the same moving average function). Thus, tail-down models allow for correlation between variables at all sites on the same network, regardless of whether they are flow-connected. The flow-connected portion of a tail-down covariance function is a function of stream distance between locations, but the flow-unconnected portion is a function of the two stream distances to the common junction. No weighting by flow (or anything else) is associated with tail-down models. Because they allow for positive correlation among both flow-connected and flow-unconnected sites on the same network, tail-down models would appear to be more well-suited than tail-up models for counts of fish, which can move both upstream and downstream.

The covariance structure of stream networks may also be a mixture of tail-up or tail-down. However, the horizon of this paper is limited to pure tail-up or pure tail-down models.

1.2 Torgegram Components

Very recently, Zimmerman and Ver Hoef (2017) introduced a graphical tool for informal variography on stream networks called the Torgegram, which is an assemblage of three empirical semivariograms, each one relevant to a particular combination of flow-connectedness and model type (tail-up/tail-down). Each of these is now described further.

1.2.1 The Flow-Unconnected Stream-Distance (FUSD) Semivariogram

As its name suggests, this empirical semivariogram is computed from only those site-pairs that are flow-unconnected and, for such pairs, is a function of total stream distance only. It is defined formally as

$$\widehat{\gamma}_{FUSD}(h_k) = \frac{1}{2N(\mathcal{U}_k)} \sum_{(s_i,t_j)\in\mathcal{U}_k} \left[Y(s_i) - Y(t_j)\right]^2, \text{ for } k = 1, 2, ..., K_{\mathcal{U}},$$

where $\mathcal{U}_k = \{(s_i, t_j) : d(s_i, t_j) \in \mathcal{H}_k, U_i \cap U_j = \emptyset\}$ is a partition of the total stream distances into bins, h_k is a representative distance within $\mathcal{U}_k$, $N(\mathcal{U}_k)$ is the number of distinct site-pairs in $\mathcal{U}_k$, and $K_{\mathcal{U}}$ is the number of stream-distance bins for those site-pairs. If $Y(\cdot)$ is pure tail-up, then $\widehat{\gamma}_{FUSD}(\cdot)$ is unbiased (apart from a "blurring" effect due to binning similar but unequal stream distances) for the flow-unconnected portion of its semivariogram, which in this case is "flat," i.e., a constant function. On the other hand, if $Y(\cdot)$ is pure tail-down or a mixture of tail-up and tail-down, then the flow-unconnected portion of its semivariogram may be a function not of total stream distance but of the two stream distances from sites within a site-pair to their common junction, in which case $\widehat{\gamma}_{FUSD}(\cdot)$ may not be fully relevant, i.e., what it intends to estimate doesn't exist. An exception occurs if the tail-down component has an exponential semivariogram; in this case the flow-unconnected portion of its semivariogram is a function of total stream distance only, hence $\widehat{\gamma}_{FUSD}(\cdot)$ remains unbiased for it (apart from blurring).

1.2.2 The Flow-Unconnected Distances-to-Common-Junction (FUDJ) Semivariogram

The second component semivariogram of the Torgegram, like the first, is computed using only those site-pairs that are flow-unconnected, but it is not a function of total stream distance. Rather, it is a function of the two stream distances from each site in the pair to their common junction. Let $\{\mathcal{J}_k : k = 1, 2, ..., K_{\mathcal{J}}\}$ be the bins of a partition of the stream distances to common junction that occur among the flow-unconnected site-pairs, let $N(\mathcal{J}_k, \mathcal{J}_l)$ be the number of such site-pairs for which one site's distance to common junction lies in $\mathcal{J}_k$ and the other's lies in $\mathcal{J}_l$, and let j_k be a representative distance within $\mathcal{J}_k$. Without loss of generality, assume that $j_k \leq j_l$. Then we define the flow-unconnected distances-to-common-junction empirical semivariogram as

$$\widehat{\gamma}_{FUDJ}(j_k, j_l) = \frac{1}{2N(\mathcal{J}_k, \mathcal{J}_l)} \sum_{s_i \in \mathcal{J}_k, t_j \in \mathcal{J}_l} [Y(s_i) - Y(t_j)]^2\,, \text{ for } k \leq l = 1, 2, ..., K_{\mathcal{J}},$$

Regardless of whether $Y(\cdot)$ is pure tail-down, pure tail-up, or a mixture thereof, $\widehat{\gamma}_{FUDJ}(\cdot,\cdot)$ is unbiased (apart from blurring) for the flow-unconnected portion of its semivariogram. This is its advantage over $\widehat{\gamma}_{FUSD}(\cdot)$, which (as noted previously) is not fully relevant unless $Y(\cdot)$ is pure tail-up or its tail-down component has an exponential semivariogram. Note, however, that $\widehat{\gamma}_{FUDJ}(\cdot,\cdot)$ is a function of two distances rather than one.

1.2.3 The Flow-Connected Stream-Distance (FCSD) Semivariogram

The FCSD empirical semivariogram is based on stream distance only but differs from the FUSD empirical semivariogram by being computed from site-pairs that are flow-connected rather than flow-unconnected. Thus it is defined as

$$\widehat{\gamma}_{FCSD}(h_k) = \frac{1}{2N(\mathcal{C}_k)} \sum_{(s_i, t_j) \in \mathcal{C}_k} [Y(s_i) - Y(t_j)]^2\,, \text{ for } k = 1, 2, ..., K_{\mathcal{C}},$$

where $\mathcal{C}_k = \{(s_i, t_j) : d(s_i, t_j) \in \mathcal{H}_k, U_i \cap U_j \neq \emptyset\}$, h_k is a representative distance within $\mathcal{C}_k$, $N(\mathcal{C}_k)$ is the number of distinct site-pairs in $\mathcal{C}_k$, and $K_{\mathcal{C}}$ is the number of stream-distance bins for those site-pairs. If $Y(\cdot)$ is pure tail-down, then $\widehat{\gamma}_{FCSD}(\cdot)$ is unbiased (apart from blurring) for the flow-connected portion of its semivariogram. If, however, $Y(\cdot)$ is pure tail-up or a tail-up/tail-down mixture, $\widehat{\gamma}_{FCSD}(\cdot)$ is not fully relevant because in those cases the flow-connected semivariogram is a function of not merely stream distance, but of stream distance and the spatial weights.

The remaining sections of this report are arranged as follows: Section 2 exhibits the construction of T_{td}^m under a regular rooted binary tree network with q sites per segment and m-$(2m{+}1/q)$-dependence and presents simulation results to show the nice behaviors when number of levels of stream network increases. Section 3 extends the test to data generated by absolutely summable covariance functions. For conciseness of the manuscript, proofs of all theorems in subsequent sections are elaborated in an appendix.

2 Testing Tail-Down Model under m-$(2m{+}1/q)$-Dependence

2.1 Methodology of Test for Tail-Down Model

Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a Gaussian random field on a regular rooted binary tree stream network, which contains segments of unit length and q data points per segment. $\{Z_{ij}\}$ is second-order stationary in the network. A is defined as the collection of segment indices, which is indexed in the fashion shown in Figure 2, while j is the index of points on the same segment, with smaller number closer to the outlet. Any adjacent points are $1/q$ units away. The number of levels of stream network, denoted as l_{sn}, is defined as the distance between the lower end of the most downstream segment and the upper end of the most upstream segment. It can be shown that $l_{sn} = \log_2(|A| + 1)$.

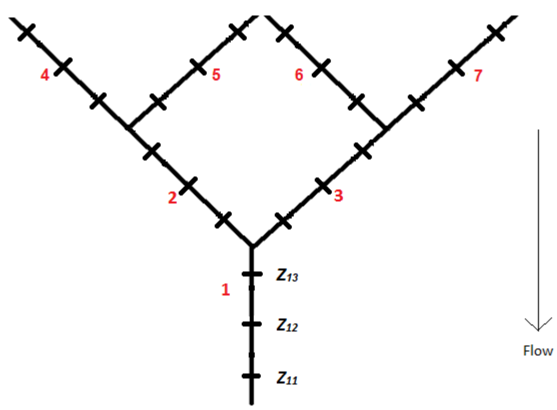


Figure 2: Stream network structure of regular rooted binary tree with $q = 3$

Assume that $\{Z_{ij}\}$ is second-order stationary on the network, and marginally $Z_{ij} \sim N(\mu, \sigma^2)$. Denote operations $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Assume that $\{Z_{ij}\}$ has tail-down covariance functions, which are specified in Section 1.1, as

$$C(Z_{ij}, Z_{st}) = \begin{cases} C_{fc}(h) & \text{if segment } i \text{ and } s \text{ are flow-connected,} \\ C_{fu}(a \wedge b, a \vee b) & \text{if segment } i \text{ and } s \text{ are flow-unconnected,} \end{cases}$$

where $h = d(Z_{ij}, Z_{st})$ is the stream distance between Z_{ij} and Z_{st}, and a, b are the distances-to-common-junction of Z_{ij} and Z_{st}. Moreover, $C_{fc}(0) = C_{fu}(0, 0) = \sigma^2$.

Definition 2.1.1. *Let $\{Z_{ij} : i \in A, j = 1, 2..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the network and covariance functions as previously specified. Then, $\{Z_{ij}\}$ is said to have m-$(2m+1/q)$-dependent tail-down (covariance) functions if there exists $m \in \{x/q : x \in \mathbb{N}_+\}$ such that*

$$\begin{cases} C_{fc}(x) > 0 & \text{if } 0 \leq x \leq m, \\ C_{fc}(x) = 0 & \text{if } x \geq m + 1/q, \\ C_{fu}(y, x) > 0 & \text{if } 0 \leq y < x \leq m, \\ C_{fu}(y, x) = 0 & \text{if } 0 \leq y < x \text{ and } x \geq m + 1/q. \end{cases}$$

By Definition 2.1.1, if $\{Z_{ij}\}$ has m-$(2m+1/q)$-dependent tail-down covariance functions, the covariance vanishes beyond at most m units (mq sites) for flow-connected pairs and $2m+1/q$ units ($2mq+1$ sites) for flow-unconnected pairs. Define the set of adjacent flow-connected segment pairs as

$$B = \{(i, s) : i < s, \text{ segment } i \text{ and } s \text{ are adjacent and flow-connected}\}.$$

By aforementioned segment numbering scheme, it can be shown that

$$B = \{(i, s) : s = 2i \text{ or } s = 2i + 1, i \in A, s \in A\}, \text{ and } |B| = |A| - 1.$$

Section 1.2 has introduced empirical FCSD and FUDJ semivariograms. Here we define the theoretical semivariograms in this network and treat empirical semivariograms as method of moment estimates of theoretical ones. The *theoretical FCSD semivariogram* is defined as

$$\gamma_c(h) = \frac{1}{2}\text{Var}\left(Z_{ij} - Z_{st}\right) = C_{fc}(0) - C_{fc}(h) = \sigma^2 - C_{fc}(h),$$

where Z_{ij} and Z_{st} are arbitrary flow-connected sites such that $d(Z_{ij}, Z_{st}) = h$. Similarly, the *theoretical FUDJ semivariogram* is defined as

$$\begin{aligned}\gamma_u\left(a \wedge b, a \vee b\right) &= \frac{1}{2}\text{Var}\left(Z_{ij} - Z_{st}\right) \\ &= C_{fu}(0,0) - C_{fu}(a \wedge b, a \vee b) = \sigma^2 - C_{fu}(a \wedge b, a \vee b),\end{aligned}$$

where Z_{ij} and Z_{st} are arbitrary flow-unconnected sites such that their distances-to-common-junction are a and b. Similar to FCSD subsemivariograms introduced in Section 2.1, two versions of *theoretical FCSD subsemivariograms* ($p = 0, 1$) at $h = 1/q$ are defined as

$$\begin{aligned}&\text{Type-0: } \gamma_0(1/q) = \frac{1}{2}\text{Var}\left(Z_{ij} - Z_{i(j+1)}\right), \text{ for any } i \in A \text{ and } j \in \{1, 2, ..., q-1\}, \\ &\text{Type-1: } \gamma_1(1/q) = \frac{1}{2}\text{Var}\left(Z_{iq} - Z_{s1}\right), \text{ for any } (i, s) \in B.\end{aligned}$$

Under the pure tail-down dependence setting, $\gamma_0(1/q) = \gamma_1(1/q) = \sigma^2 - C_{fc}(1/q)$, since $C_{fc}(\cdot)$ is solely a function of stream distance and doesn't involve spatial weights. Hence, a test can be constructed to evaluate the estimate of $\gamma_0(1/q) - \gamma_1(1/q)$. FCSD Type-0 and Type-1 subsemivariograms in this network can be specified as

$$\widehat{\gamma}_0(1/q) = \frac{1}{2(q-1)|A|}\sum_{i=1}^{|A|}\sum_{j=1}^{q-1}\left(Z_{ij} - Z_{i(j+1)}\right)^2, \tag{2}$$

$$\begin{aligned}\widehat{\gamma}_1(1/q) &= \frac{1}{2|B|}\sum_{(i,s)\in B}\left(Z_{iq} - Z_{s1}\right)^2 \\ &= \frac{1}{2(|A|-1)}\sum_{i=1}^{(|A|-1)/2}\left[\left(Z_{iq} - Z_{(2i)1}\right)^2 + \left(Z_{iq} - Z_{(2i+1)1}\right)^2\right].\end{aligned} \tag{3}$$

Assume that $\{Z_{ij}\}$ has m-($2m$+$1/q$)-dependent tail-down covariance functions on this network. Define

$$\underline{\widehat{\gamma}} = \begin{bmatrix} \widehat{\gamma}_0(1/q) \\ \widehat{\gamma}_1(1/q) \end{bmatrix}, \quad \underline{\gamma} = \begin{bmatrix} \gamma_0(1/q) \\ \gamma_1(1/q) \end{bmatrix}.$$

Assume that, as the number of stream levels l_{sn} approaches infinity, the limiting distribution of $\sqrt{|A|}(\underline{\widehat{\gamma}} - \underline{\gamma})$ is multivariate normal with mean $\underline{0}$ and some covariance $\mathbf{\Sigma}$, which will be proved in subsequent subsections. Define the elements of $\mathbf{\Sigma}$, i.e., the asymptotic variances (and covariances) of FCSD subsemivariograms, as (with superscript m for m-$(2m+1/q)$-dependence)

$$\sigma_{11}^m = \lim_{l_{sn}\to\infty} |A|\text{Var}[\widehat{\gamma}_0(1/q)],$$
$$\sigma_{22}^m = \lim_{l_{sn}\to\infty} |A|\text{Var}[\widehat{\gamma}_1(1/q)],$$
$$\sigma_{12}^m = \lim_{l_{sn}\to\infty} |A|\text{Cov}[\widehat{\gamma}_0(1/q), \widehat{\gamma}_1(1/q)],$$

and $\widehat{\sigma}_{11}^m$, $\widehat{\sigma}_{22}^m$, $\widehat{\sigma}_{12}^m$ their consistent estimators as $l_{sn} \to \infty$. Let

$$T_{td}^m = \frac{|A|\left[\widehat{\gamma}_0(1/q) - \widehat{\gamma}_1(1/q)\right]^2}{\widehat{\sigma}_{11}^m + \widehat{\sigma}_{22}^m - 2\widehat{\sigma}_{12}^m}. \tag{4}$$

When $\{Z_{ij}\}$ has an m-$(2m+1/q)$-dependent tail-down model, the null hypothesis $H_0 : \gamma_0(1/q) = \gamma_1(1/q)$ is true. Then, by the Slutsky's Theorem,

$$T_{td}^m = \frac{\left\{\sqrt{|A|}\left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] - \sqrt{|A|}\left[\widehat{\gamma}_1(1/q) - \gamma_1(1/q)\right]\right\}^2}{\widehat{\sigma}_{11}^m + \widehat{\sigma}_{22}^m - 2\widehat{\sigma}_{12}^m} \xrightarrow{d} \chi_1^2 \text{ as } l_{sn} \to \infty.$$

Hence when $T_{td}^m > \chi_{1,\alpha}^2$, reject $H_0 : \gamma_0(1/q) = \gamma_1(1/q)$, i.e., reject that $\{Z_{ij}\}$ has the covariance structure of an m-$(2m+1/q)$-dependent tail-down model.

2.2 Asymptotic Variances and Covariances

Theorem 2.2.1. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the stream network and covariance functions as previously defined. Then*

$$\begin{aligned}\sigma_{11}^m =& \lim_{l_{sn}\to\infty} |A|\text{Var}\left[\widehat{\gamma}_0(1/q)\right]\\ =&\frac{1}{2(q-1)}\left[2\gamma_0(1/q)\right]^2 + \frac{q-2}{(q-1)^2}\left[2\gamma_0(1/q) - \gamma_c(2/q)\right]^2 + \frac{(q-3)\vee 0}{(q-1)^2}\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2\\ &+ \frac{1}{(q-1)^2}\sum_{k=3}^{(qm+1)\wedge(q-2)} (q-k-1)\Big\{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]\Big\}^2\\ &+ \frac{1}{(q-1)^2}\left[\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(q-1)^2} \sum_{l=3}^{2q-2} [q-1-(l-q) \vee (q-l)] \Big\{ \gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q] \Big\}^2 \\
&+ \frac{1}{(q-1)^2} \sum_{k=2}^{\lceil m \rceil} \sum_{l=q(k-1)+2}^{q(k+1)-2} [q-1-(l-qk) \vee (qk-l)] \Big\{ \gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q] \Big\}^2 \\
&+ \frac{1}{(q-1)^2} \sum_{k=0}^{\lfloor m \rfloor} 2^{k-1} \sum_{u=qk}^{q(k+1)-2} \Big\{ \gamma_u[(2u+1)/2q, (2u+1)/2q] - 2\gamma_u[(2u+1)/2q, (2u+3)/2q] \\
&\qquad + \gamma_u[(2u+3)/2q, (2u+3)/2q] \Big\}^2 \\
&+ \frac{1}{(q-1)^2} \sum_{k=0}^{\lfloor m \rfloor} 2^{k} \sum_{u=qk+1}^{q(k+1)-2} \sum_{v=qk}^{u-1} \Big\{ \gamma_u[(2v+1)/2q, (2u+1)/2q] - \gamma_u[(2v+3)/2q, (2u+1)/2q] \\
&\qquad - \gamma_u[(2v+1)/2q, (2u+3)/2q] + \gamma_u[(2v+3)/2q, (2u+3)/2q] \Big\}^2 \\
&+ \frac{1}{(q-1)^2} \sum_{k=1}^{\lfloor m \rfloor} \sum_{l=0}^{k-1} 2^{l} \sum_{u=qk}^{q(k+1)-2} \sum_{v=ql}^{q(l+1)-2} \Big\{ \gamma_u[(2v+1)/2q, (2u+1)/2q] - \gamma_u[(2v+3)/2q, (2u+1)/2q] \\
&\qquad - \gamma_u[(2v+1)/2q, (2u+3)/2q] + \gamma_u[(2v+3)/2q, (2u+3)/2q] \Big\}^2 .
\end{aligned}
$$

Proof. See Appendix A.1.1. □

Theorem 2.2.2. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the stream network and covariance functions as previously defined. Then*

$$
\begin{aligned}
\sigma_{22}^m =& \lim_{l_{sn} \to \infty} |A| \mathrm{Var}\,[\widehat{\gamma}_1(1/q)] \\
=& \frac{1}{2}\,[2\gamma_1(1/q)]^2 + \frac{1}{2}\,[2\gamma_1(1/q) - \gamma_u(1/2q, 1/2q)]^2 + \mathrm{I}\{q=2\} \cdot \mathrm{I}\Big\{ \lfloor m + 1/q \rfloor \geq 1 \Big\} \cdot [\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)]^2 \\
&+ \mathrm{I}\{q \geq 3\} \cdot \mathrm{I}\Big\{ \lfloor m + 1/q \rfloor \geq 1 \Big\} \cdot [\gamma_c(1-1/q) - 2\gamma_c(1) + \gamma_c(1+1/q)]^2 \\
&+ \sum_{k=2}^{\lfloor m+1/q \rfloor} [\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)]^2 \\
&+ \sum_{k=1}^{\lfloor m+1/q \rfloor} [\gamma_c(k) - \gamma_c(k+1/q) - \gamma_u(1/2q, k-1/2q) + \gamma_u(1/2q, k+1/2q)]^2 \\
&+ \sum_{k=1}^{\lfloor m+1/q \rfloor} 2^{k-1} [\gamma_u(k-1/2q, k-1/2q) - 2\gamma_u(k-1/2q, k+1/2q) + \gamma_u(k+1/2q, k+1/2q)]^2 \\
&+ \sum_{k=2}^{\lfloor m+1/q \rfloor} \sum_{l=1}^{k-1} 2^{l} \big[\gamma_u(l-1/2q, k-1/2q) - \gamma_u(l+1/2q, k-1/2q) \\
&\qquad - \gamma_u(l-1/2q, k+1/2q) + \gamma_u(l+1/2q, k+1/2q) \big]^2 .
\end{aligned}
$$

Proof. See Appendix A.1.2. □

Theorem 2.2.3. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the stream network and covariance functions as* *previously defined. Then*

$$
\begin{aligned}
\sigma_{12}^{m} =& \lim_{l_{sn}\to\infty} |A| \mathrm{Cov}\left[\widehat{\gamma}_0(1/q), \widehat{\gamma}_1(1/q)\right] \\
=& \frac{1}{q-1}\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2 + \mathrm{I}\{q\geq 3\}\cdot\frac{1}{q-1}\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2 \\
&+ \frac{1}{q-1}\sum_{l=3}^{q-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2 \\
&+ \frac{1}{2(q-1)}\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2 \\
&+ \mathrm{I}\{q\geq 3\}\cdot\frac{1}{2(q-1)}\left[\gamma_c(2/q)-\gamma_c(3/q)-\gamma_u(1/2q,3/2q)+\gamma_u(1/2q,5/2q)\right]^2 \\
&+ \frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2 \\
&+ \frac{1}{2(q-1)}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{k=2}^{\lceil m\rceil}\sum_{l=1}^{k-1}2^{l-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2lq-1)/2q,(2u+1)/2q]-\gamma_u[(2lq+1)/2q,(2u+1)/2q] \\
&\qquad -\gamma_u[(2lq-1)/2q,(2u+3)/2q]+\gamma_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{l=1}^{\lceil m\rceil}\sum_{k=1}^{l}2^{k-2}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2u+1)/2q,(2lq-1)/2q]-\gamma_u[(2u+1)/2q,(2lq+1)/2q] \\
&\qquad -\gamma_u[(2u+3)/2q,(2lq-1)/2q]+\gamma_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2.
\end{aligned}
$$

Proof. See Appendix A.1.3. □

2.3 Asymptotic Normality

Theorem 2.3.1. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the stream network and covariance functions as previously defined. Then the random vector*

$$
\sqrt{|A|}\begin{bmatrix} \widehat{\gamma}_0(1/q)-\gamma_0(1/q) \\ \widehat{\gamma}_1(1/q)-\gamma_1(1/q)\end{bmatrix} \xrightarrow{d} N\left(\underline{0}, \mathbf{\Sigma}^m\right),
$$

as $l_{sn} \to \infty$, *with*

$$\mathbf{\Sigma}^m = \begin{bmatrix} \sigma_{11}^m & \sigma_{12}^m \\ \sigma_{12}^m & \sigma_{22}^m \end{bmatrix}$$

and the elements as specified in Theorem 2.2.1 – 2.2.3.

Proof. See Appendix A.2. □

2.4 Consistency of $\widehat{\mathbf{\Sigma}}^m$

Theorem 2.4.1. *Let* $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ *be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the stream network and covariance functions as previously defined. Let*

$$\begin{aligned}
\widehat{\sigma}_{11}^m =& \frac{1}{2(q-1)}\left[2\widehat{\gamma}_0(1/q)\right]^2 + \frac{q-2}{(q-1)^2}\left[2\widehat{\gamma}_0(1/q) - \widehat{\gamma}_c(2/q)\right]^2 + \frac{(q-3)\vee 0}{(q-1)^2}\left[\widehat{\gamma}_0(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=3}^{(qm+1)\wedge(q-2)}(q-k-1)\Big\{\widehat{\gamma}_c[(k-1)/q] - 2\widehat{\gamma}_c(k/q) + \widehat{\gamma}_c[(k+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\left[\widehat{\gamma}_1(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\widehat{\gamma}_c[(l-1)/q] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=2}^{\lceil m\rceil}\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\widehat{\gamma}_c[(l-1)/q] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\lfloor m\rfloor}2^{k-1}\sum_{u=qk}^{q(k+1)-2}\Big\{\widehat{\gamma}_u[(2u+1)/2q,(2u+1)/2q] - 2\widehat{\gamma}_u[(2u+1)/2q,(2u+3)/2q] \\
&\qquad + \widehat{\gamma}_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\lfloor m\rfloor}2^{k}\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\widehat{\gamma}_u[(2v+1)/2q,(2u+1)/2q] - \widehat{\gamma}_u[(2v+3)/2q,(2u+1)/2q] \\
&\qquad - \widehat{\gamma}_u[(2v+1)/2q,(2u+3)/2q] + \widehat{\gamma}_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=0}^{k-1}2^{l}\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\widehat{\gamma}_u[(2v+1)/2q,(2u+1)/2q] - \widehat{\gamma}_u[(2v+3)/2q,(2u+1)/2q] \\
&\qquad - \widehat{\gamma}_u[(2v+1)/2q,(2u+3)/2q] + \widehat{\gamma}_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2,
\end{aligned}$$

$$\begin{aligned}
\widehat{\sigma}_{22}^m =& \frac{1}{2}\left[2\widehat{\gamma}_1(1/q)\right]^2 + \frac{1}{2}\left[2\widehat{\gamma}_1(1/q) - \widehat{\gamma}_u(1/2q, 1/2q)\right]^2 + \mathrm{I}\{q=2\}\cdot\left[\widehat{\gamma}_0(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)\right]^2 \\
& + \mathrm{I}\{q\geq 3\}\cdot\left[\widehat{\gamma}_c(1-1/q) - 2\widehat{\gamma}_c(1) + \widehat{\gamma}_c(1+1/q)\right]^2 + \sum_{k=2}^{\lfloor m+1/q\rfloor}\left[\widehat{\gamma}_c(k-1/q) - 2\widehat{\gamma}_c(k) + \widehat{\gamma}_c(k+1/q)\right]^2 \\
& + \sum_{k=1}^{\lfloor m+1/q\rfloor}\left[\widehat{\gamma}_c(k) - \widehat{\gamma}_c(k+1/q) - \widehat{\gamma}_u(1/2q, k-1/2q) + \widehat{\gamma}_u(1/2q, k+1/2q)\right]^2 \\
& + \sum_{k=1}^{\lfloor m+1/q\rfloor} 2^{k-1}\left[\widehat{\gamma}_u(k-1/2q, k-1/2q) - 2\widehat{\gamma}_u(k-1/2q, k+1/2q) + \widehat{\gamma}_u(k+1/2q, k+1/2q)\right]^2 \\
& + \sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1} 2^l\big[\widehat{\gamma}_u(l-1/2q, k-1/2q) - \widehat{\gamma}_u(l+1/2q, k-1/2q) \\
& \qquad - \widehat{\gamma}_u(l-1/2q, k+1/2q) + \widehat{\gamma}_u(l+1/2q, k+1/2q)\big]^2, \\
\widehat{\sigma}_{12}^m =& \frac{1}{q-1}\left[\widehat{\gamma}_0(1/q) + \widehat{\gamma}_1(1/q) - \widehat{\gamma}_c(2/q)\right]^2 + \mathrm{I}\{q\geq 3\}\cdot\frac{1}{q-1}\left[\widehat{\gamma}_0(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)\right]^2 \\
& + \frac{1}{2(q-1)}\sum_{l=3}^{q-1}\Big\{\widehat{\gamma}_c\left[(l-1)/q\right] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c\left[(l+1)/q\right]\Big\}^2 \\
& + \frac{1}{q-1}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\widehat{\gamma}_c\left[(l-1)/q\right] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c\left[(l+1)/q\right]\Big\}^2 \\
& + \frac{1}{2(q-1)}\left[\widehat{\gamma}_1(1/q) - \widehat{\gamma}_c(2/q) - \widehat{\gamma}_u(1/2q, 1/2q) + \widehat{\gamma}_u(1/2q, 3/2q)\right]^2 \\
& + \mathrm{I}\{q\geq 3\}\cdot\frac{1}{2(q-1)}\left[\widehat{\gamma}_c(2/q) - \widehat{\gamma}_c(3/q) - \widehat{\gamma}_u(1/2q, 3/2q) + \widehat{\gamma}_u(1/2q, 5/2q)\right]^2 \\
& + \frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\widehat{\gamma}_c[(l+1)/q] - \widehat{\gamma}_c[(l+2)/q] - \widehat{\gamma}_u[1/2q, (2l+1)/2q] + \widehat{\gamma}_u[1/2q, (2l+3)/2q]\Big\}^2 \\
& + \frac{1}{2(q-1)}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=qk}^{q(k+1)-2}\Big\{\widehat{\gamma}_c[(l+1)/q] - \widehat{\gamma}_c[(l+2)/q] - \widehat{\gamma}_u[1/2q, (2l+1)/2q] + \widehat{\gamma}_u[1/2q, (2l+3)/2q]\Big\}^2 \\
& + \frac{1}{q-1}\sum_{k=2}^{\lceil m\rceil}\sum_{l=1}^{k-1} 2^{l-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\widehat{\gamma}_u[(2lq-1)/2q, (2u+1)/2q] - \widehat{\gamma}_u[(2lq+1)/2q, (2u+1)/2q] \\
& \qquad - \widehat{\gamma}_u[(2lq-1)/2q, (2u+3)/2q] + \widehat{\gamma}_u[(2lq+1)/2q, (2u+3)/2q]\Big\}^2 \\
& + \frac{1}{q-1}\sum_{l=1}^{\lceil m\rceil}\sum_{k=1}^{l} 2^{k-2}\sum_{u=q(k-1)}^{qk-2}\Big\{\widehat{\gamma}_u[(2u+1)/2q, (2lq-1)/2q] - \widehat{\gamma}_u[(2u+1)/2q, (2lq+1)/2q] \\
& \qquad - \widehat{\gamma}_u[(2u+3)/2q, (2lq-1)/2q] + \widehat{\gamma}_u[(2u+3)/2q, (2lq+1)/2q]\Big\}^2,
\end{aligned}$$

where $\widehat{\gamma}_c(h)$ and $\widehat{\gamma}_u(a,b)$ are regular method of moment estimators of FCSD and FUDJ semivariograms with different lags or distances to common junctions. Then $\widehat{\sigma}_{11}^m \xrightarrow{p} \sigma_{11}^m$, $\widehat{\sigma}_{12}^m \xrightarrow{p} \sigma_{12}^m$ and $\widehat{\sigma}_{22}^m \xrightarrow{p} \sigma_{22}^m$ as $l_{sn}\to\infty$.

Proof. From Theorem 2.3.1, $\sqrt{|A|}\left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] \xrightarrow{d} N(0, \sigma_{11}^m)$ and $\sqrt{|A|}\left[\widehat{\gamma}_1(1/q) - \gamma_1(1/q)\right] \xrightarrow{d} N(0, \sigma_{22}^m)$ as $l_{sn}\to\infty$. Hence, $\widehat{\gamma}_0(1/q) - \gamma_0(1/q) \xrightarrow{p} 0$ and $\widehat{\gamma}_1(1/q) - \gamma_1(1/q) \xrightarrow{p} 0$ as $l_{sn}\to\infty$. If it can be shown that $\widehat{\gamma}_c(h)$ and $\widehat{\gamma}_u(a,b)$ are consistent estimators of $\gamma_c(h)$ and $\gamma_u(a,b)$, as $l_{sn}\to\infty$, then $\widehat{\sigma}_{11}^m$, $\widehat{\sigma}_{12}^m$ and $\widehat{\sigma}_{22}^m$ would be consistent estimators of the corresponding elements of $\mathbf{\Sigma}^m$ by the Continuous

Mapping Theorem.

The proof of the consistency of $\widehat{\gamma}_c(1)$ is elaborated here, and the consistency of $\widehat{\gamma}_c(h)$ and $\widehat{\gamma}_u(a,b)$ with arbitrary lags can be proved in similar fashion. The estimator of $\gamma_c(1)$ is

$$\widehat{\gamma}_c(1) = \frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \sum_{j=1}^{q} \left[\left(Z_{ij} - Z_{(2i)j}\right)^2 + \left(Z_{ij} - Z_{(2i+1)j}\right)^2 \right].$$

Define $G_i = \sum_{j=1}^{q} \left[\left(Z_{ij} - Z_{(2i)j}\right)^2 + \left(Z_{ij} - Z_{(2i+1)j}\right)^2 \right]$. Since $\widehat{\gamma}_c(1)$ is a method of moments estimator, it is unbiased for $\gamma_c(1)$. Since $\{Z_{ij}\}$ is m-$(2m+1/q)$-dependent, for for every fixed i the number of non-zero $\mathrm{Cov}(G_i, G_s)$ is finite and bounded by a common integer. Define this upper bound as N_G. Since $\mathrm{E}\left(Z_{ij}^4\right) < \infty$ for a normal distribution, by the Cauchy-Schwarz Inequality, for every i, $\mathrm{Var}\left(G_i\right) \leq M$ for some large M. Thus, again by the Cauchy-Schwarz Inequality and the limited range of dependence of $\{G_i\}$, for fixed i,

$$\sum_{s=1}^{(|A|-1)/2} \mathrm{Cov}\left(G_i, G_s\right) \leq N_G \left[\frac{1}{2}\mathrm{Var}\left(G_i\right) + \frac{1}{2}\mathrm{Var}\left(G_1\right)\right] \leq N_G \left(\frac{1}{2}M + \frac{1}{2}M\right) = MN_G.$$

Thus, as $l_{sn} \to \infty$, which implies that $|A| \to \infty$,

$$\begin{aligned} \mathrm{Var}\left[\widehat{\gamma}_c(1)\right] &= \frac{1}{4(|A|-1)^2} \sum_{i=1}^{(|A|-1)/2} \sum_{s=1}^{(|A|-1)/2} \mathrm{Cov}\left(G_i, G_s\right) \\ &\leq \frac{1}{4(|A|-1)^2} \sum_{i=1}^{(|A|-1)/2} MN_G = \frac{MN_G}{8(|A|-1)} \to 0, \end{aligned}$$

which implies $\lim_{l_{sn}\to\infty} \mathrm{Var}\left[\widehat{\gamma}_c(1)\right] = 0$. Recall that $\widehat{\gamma}_c(1)$ is unbiased of $\gamma_c(1)$, then for any fixed $\epsilon > 0$,

$$\limsup_{l_{sn}\to\infty} \mathrm{P}\left[\left|\widehat{\gamma}_c(1) - \gamma_c(1)\right| > \epsilon\right] \leq \limsup_{l_{sn}\to\infty} \frac{\mathrm{Var}\left[\widehat{\gamma}_c(1)\right]}{\epsilon^2} = 0,$$

i.e., $\widehat{\gamma}_c(1) - \gamma_c(1) \xrightarrow{p} 0$ as $l_{sn} \to \infty$. As aforementioned, the consistency of $\widehat{\gamma}_c(h)$ and $\widehat{\gamma}_u(a,b)$ with arbitrary fixed lags can be proved in a similar fashion. This completes the proof. □

2.5 Simulation

A simulation study was conducted to evaluate the test of tail-down covariance structure using T_{td}^m in Section 2.1. Six models are used to generate stream network data. For simplicity of expression, the abbreviations TU and TD stand for "tail-up" and "tail-down", respectively. The covariance functions of the six models are:

1. Pure TD Linear with Sill Model (TD Lin.)

   Assume $b \geq a \geq 0$,

$$C_{td}(h, a, b) = \begin{cases} \theta_v(1 - h/\theta_r)I(h/\theta_r \leq 1) & \text{flow connected,} \\ \theta_v(1 - b/\theta_r)I(b/\theta_r \leq 1) & \text{flow unconnected.} \end{cases}$$

2. Pure TD Spherical Model (TD Sph.)

   Assume $b \geq a \geq 0$,

$$C_{td}(h, a, b) = \begin{cases} \theta_v(1 - 3h/2\theta_r + h^3/2\theta_r^3)I(h/\theta_r \leq 1) & \text{flow connected,} \\ \theta_v(1 - 3a/2\theta_r + b/2\theta_r)(1 - b/\theta_r)^2 I(b/\theta_r \leq 1) & \text{flow unconnected.} \end{cases}$$

3. Pure TD Exponential Model (TD Exp.)

$$C_{td}(h, a, b) = \begin{cases} \theta_v \exp\{-h/\theta_r\} & \text{flow connected,} \\ \theta_v \exp\{-(a + b)/\theta_r\} & \text{flow unconnected.} \end{cases}$$

4. Pure TU Exponential Model (TU Exp.)

$$C_{tu}\left[h = d(Z_{is}, Z_{jt})\right] = \begin{cases} \pi_{ij} \cdot \theta_v \exp\{-h/\theta_r\} & \text{flow connected,} \\ 0 & \text{flow unconnected.} \end{cases}$$

5. Pure TU Linear with Sill Model (TU Lin.)

$$C_{tu}\left[h = d(Z_{is}, Z_{jt})\right] = \begin{cases} \pi_{ij} \cdot \theta_v (1 - h/\theta_r) I(h/\theta_r \leq 1) & \text{flow connected,} \\ 0 & \text{flow unconnected.} \end{cases}$$

6. Pure TU Spherical Model (TU Sph.)

$$C_{tu}\left[h = d(Z_{is}, Z_{jt})\right] = \begin{cases} \pi_{ij} \cdot \theta_v (1 - 3h/2\theta_r + h^3/2\theta_r^3) I(h/\theta_r \leq 1) & \text{flow connected,} \\ 0 & \text{flow unconnected.} \end{cases}$$

$\theta_v = 1$ for all six models. For comparable $\gamma_0(1/q)$ across six models, θ_r are chosen at different levels such that the range parameter of TD linear with sill model, $\theta_{r,Lin}$, vary among $(1.0, 1.5, 2.0, 2.5)$. The number of points per segment, q, is chosen from $(2, 3, 4)$. Since m is a function of q, $\theta_{r,Lin}$ is used as benchmark of spatial dependence across different value of q in this simulation. Hence, the range parameters are fixed at different levels of q and m is chosen based on q and $\theta_{r,Lin}$. It can be shown that $m = \lceil q\theta_{r,Lin} - 1 \rceil / q$. The table that tabulates range parameters as well as values of m by q and $\theta_{r,Lin}$ is shown below.

	Model	$m = 1/2$	$m = 1$	$m = 3/2$	$m = 2$
		θ_r			
$q = 2$	TD Lin. & TU Lin.	1.0000	1.5000	2.0000	2.5000
	TD Exp. & TU Exp.	0.7213	1.2332	1.7380	2.2407
	TD Sph. & TU Sph.	1.4397	2.2117	2.9719	3.7276
	Model	$m = 2/3$	$m = 4/3$	$m = 5/3$	$m = 7/3$
		θ_r			
$q = 3$	TD Lin. & TU Lin.	1.0000	1.5000	2.0000	2.5000
	TD Exp. & TU Exp.	0.8221	1.3264	1.8283	2.3294
	TD Sph. & TU Sph.	1.4744	2.2334	2.9876	3.7401
	Model	$m = 3/4$	$m = 5/4$	$m = 7/4$	$m = 9/4$
		θ_r			
$q = 4$	TD Lin. & TU Lin.	1.0000	1.5000	2.0000	2.5000
	TD Exp. & TU Exp.	0.8690	1.3712	1.8722	2.3728
	TD Sph. & TU Sph.	1.4858	2.2407	2.9930	3.7444

The TD linear with sill model satisfies both m-$(2m+1/q)$-dependence and tail-down covariance structure, thus the rejection rates under this model serve as empirical sizes. The test for tail-down model is intended to test against TU models, thus the rejection rates under three TU models serve as empirical powers.

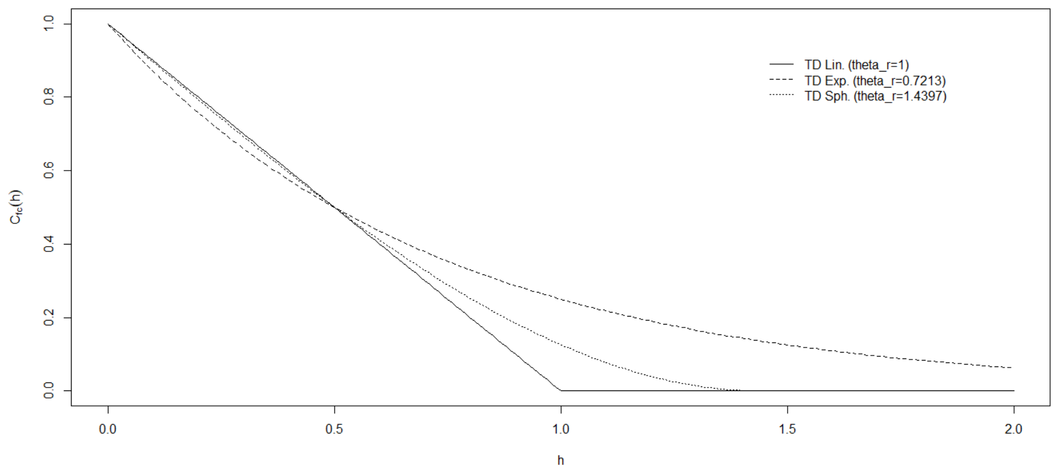


Figure 3: $C_{fc}(h)$ vs. h for three TD models with chosen θ_r's under $q = 2$ and $m = 0.5$

The TD exponential model violates m-($2m$+$1/q$)-dependence assumption because the covariance only vanishes when lag becomes infinity. The TD spherical model has finite range, but due to θ_r chosen based on different m, its spatial dependence is slightly stronger than m-($2m$+$1/q$)-dependence. Figure 3 shows the plot of C_{fc} for three TD models with $q = 2$ and $m = 0.5$. For the TD linear with sill model, C_{fc} vanishes at lag $m + 0.5 = 1$. For the TD exponential model, $C_{fc}(h) > 0$ for any $h > 0$. For the TD spherical model, C_{fc} vanishes at lag $1.4397 > m + 0.5 = 1$. Hence, these two TD models violate m-($2m$+$1/q$)-dependence assumption to different extents. The rejection rates under these two TD models would reflect how robust this test is to violation of m-($2m$+$1/q$)-dependence assumption.

Data are generated along stream networks as specified in Section 2.1, with the number of stream levels l_{sn} ranging from 3 to 8. The sample sizes at each level of l_{sn} are listed in Table 1. The rejection rates with $S = 1,000$ simulation replicate under each scenario are tabulated in Table 2 – 4.

q	# of Stream Network Levels					
	3	4	5	6	7	8
2	14	30	62	126	254	510
3	21	45	93	189	381	765
4	28	60	124	252	508	1,020

Table 1: Sample sizes by # of stream network levels and q

According to Theorem 2.4.1, $\widehat{\mathbf{\Sigma}}^m$ is a function of m, which characterizes the spatial dependence.

However, in practice, m is an unknown quantity similar to sill and range parameters. Thus, the test is conducted as follows:

Step 1. Define $H = \{h$: FC lags within the given stream network$\}$. Subset H and define the subset $H^* = \big\{h : h \in H$ and $h \leq 0.5 * \max(H)\big\}$.

Step 2. Define the threshold thd as the 80th percentile of $\{\widehat{\gamma}_c(h) : h \in H^*\}$.

Step 3. The estimate of θ_r, $\widehat{\theta}_r$, is defined as $\min\{h : \widehat{\gamma}_c(h) \geq thd$ and $h \in H^*\}$.

Step 4. The estimate of m, $\widehat{m}$, equal to $\max(\lfloor\widehat{\theta}_r - 1/q\rfloor, 1/q)$.

Step 5. Compute $T_{td}^{\widehat{m}}$ with m in (4) replaced by $\widehat{m}$, then compare $T_{td}^{\widehat{m}}$ with $\chi^2_{1,0.05}$.

	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	1/2	1.000	0.500	0.003	0.014	0.021	0.031	0.035	0.051
	TD Sph.	1/2	1.440	0.500	0.011	0.015	0.028	0.038	0.035	0.036
	TD Exp.	1/2	0.721	0.500	0.009	0.017	0.016	0.038	0.039	0.047
	TU Exp.	1/2	0.721	0.500	0.008	0.015	0.058	0.116	0.310	0.593
power	TU Lin.	1/2	1.000	0.500	0.010	0.013	0.028	0.106	0.246	0.449
	TU Sph.	1/2	1.440	0.500	0.014	0.022	0.052	0.114	0.251	0.515
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	2/2	1.500	0.333	0.008	0.039	0.047	0.045	0.043	0.040
	TD Sph.	2/2	2.212	0.333	0.007	0.024	0.037	0.044	0.042	0.053
	TD Exp.	2/2	1.233	0.333	0.013	0.021	0.028	0.033	0.025	0.036
	TU Exp.	2/2	1.233	0.333	0.016	0.035	0.132	0.362	0.723	0.952
power	TU Lin.	2/2	1.500	0.333	0.011	0.039	0.149	0.439	0.762	0.979
	TU Sph.	2/2	2.212	0.333	0.009	0.049	0.123	0.383	0.702	0.962
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	3/2	2.000	0.250		0.027	0.038	0.056	0.067	0.071
	TD Sph.	3/2	2.972	0.250		0.018	0.026	0.043	0.039	0.042
	TD Exp.	3/2	1.738	0.250		0.020	0.023	0.038	0.025	0.031
	TU Exp.	3/2	1.738	0.250		0.043	0.202	0.562	0.923	1.000
power	TU Lin.	3/2	2.000	0.250		0.067	0.280	0.562	0.919	0.997
	TU Sph.	3/2	2.972	0.250		0.053	0.237	0.556	0.918	0.998
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	4/2	2.500	0.200		0.025	0.022	0.048	0.043	0.042
	TD Sph.	4/2	3.728	0.200		0.019	0.027	0.033	0.033	0.039
	TD Exp.	4/2	2.241	0.200		0.019	0.027	0.039	0.035	0.034
	TU Exp.	4/2	2.241	0.200		0.063	0.332	0.699	0.983	1.000
power	TU Lin.	4/2	2.500	0.200		0.084	0.350	0.705	0.987	1.000
	TU Sph.	4/2	3.728	0.200		0.057	0.349	0.715	0.972	1.000

Table 2: Simulation Results of Test for Tail-Down with $q = 2$ and $S = 1,000$

	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	2/3	1.000	0.333	0.046	0.058	0.056	0.052	0.058	0.073
	TD Sph.	2/3	1.474	0.333	0.028	0.054	0.042	0.037	0.052	0.054
	TD Exp.	2/3	0.822	0.333	0.041	0.041	0.031	0.042	0.035	0.033
	TU Exp.	2/3	0.822	0.333	0.046	0.070	0.187	0.443	0.812	0.983
power	TU Lin.	2/3	1.000	0.333	0.053	0.078	0.191	0.487	0.770	0.981
	TU Sph.	2/3	1.474	0.333	0.049	0.069	0.182	0.453	0.819	0.993
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	4/3	1.500	0.222		0.047	0.054	0.055	0.052	0.043
	TD Sph.	4/3	2.233	0.222		0.038	0.048	0.034	0.040	0.039
	TD Exp.	4/3	1.326	0.222		0.039	0.031	0.027	0.035	0.030
	TU Exp.	4/3	1.326	0.222		0.098	0.345	0.754	0.981	0.999
power	TU Lin.	4/3	1.500	0.222		0.092	0.367	0.768	0.985	1.000
	TU Sph.	4/3	2.233	0.222		0.101	0.375	0.757	0.983	1.000
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	5/3	2.000	0.167		0.037	0.075	0.069	0.066	0.071
	TD Sph.	5/3	2.988	0.167		0.042	0.048	0.027	0.028	0.026
	TD Exp.	5/3	1.828	0.167		0.042	0.024	0.035	0.024	0.033
	TU Exp.	5/3	1.828	0.167		0.132	0.499	0.864	0.998	1.000
power	TU Lin.	5/3	2.000	0.167		0.130	0.520	0.884	0.999	1.000
	TU Sph.	5/3	2.988	0.167		0.121	0.509	0.869	0.995	1.000
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
					3	4	5	6	7	8
size	TD Lin.	7/3	2.500	0.133			0.069	0.036	0.036	0.049
	TD Sph.	7/3	3.740	0.133			0.047	0.026	0.026	0.023
	TD Exp.	7/3	2.329	0.133			0.044	0.027	0.027	0.031
	TU Exp.	7/3	2.329	0.133			0.616	0.913	1.000	1.000
power	TU Lin.	7/3	2.500	0.133			0.609	0.917	1.000	1.000
	TU Sph.	7/3	3.740	0.133			0.598	0.913	1.000	1.000

Table 3: Simulation Results of Test for Tail-Down with $q = 3$ and $S = 1,000$

	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
size	TD Lin.	3/4	1.000	0.250	0.045	0.090	0.079	0.057	0.069	0.058
	TD Sph.	3/4	1.486	0.250	0.032	0.062	0.042	0.044	0.039	0.036
	TD Exp.	3/4	0.869	0.250	0.030	0.061	0.066	0.035	0.031	0.042
	TU Exp.	3/4	0.869	0.250	0.032	0.083	0.322	0.689	0.981	0.999
power	TU Lin.	3/4	1.000	0.250	0.046	0.112	0.353	0.737	0.978	1.000
	TU Sph.	3/4	1.486	0.250	0.044	0.092	0.319	0.717	0.983	0.999
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
size	TD Lin.	5/4	1.500	0.167		0.042	0.062	0.045	0.044	0.041
	TD Sph.	5/4	2.241	0.167		0.033	0.048	0.042	0.028	0.025
	TD Exp.	5/4	1.371	0.167		0.037	0.044	0.029	0.023	0.031
	TU Exp.	5/4	1.371	0.167		0.120	0.518	0.897	1.000	1.000
power	TU Lin.	5/4	1.500	0.167		0.148	0.523	0.922	0.998	1.000
	TU Sph.	5/4	2.241	0.167		0.133	0.541	0.910	1.000	1.000
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
size	TD Lin.	7/4	2.000	0.125		0.053	0.084	0.077	0.067	0.061
	TD Sph.	7/4	2.993	0.125		0.022	0.047	0.023	0.016	0.022
	TD Exp.	7/4	1.872	0.125		0.047	0.051	0.027	0.029	0.018
	TU Exp.	7/4	1.872	0.125		0.161	0.653	0.940	1.000	1.000
power	TU Lin.	7/4	2.000	0.125		0.184	0.638	0.960	1.000	1.000
	TU Sph.	7/4	2.993	0.125		0.175	0.641	0.933	1.000	1.000
	Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
size	TD Lin.	9/4	2.500	0.100			0.063	0.036	0.046	0.028
	TD Sph.	9/4	3.744	0.100			0.036	0.020	0.010	0.023
	TD Exp.	9/4	2.373	0.100			0.041	0.028	0.023	0.017
	TU Exp.	9/4	2.373	0.100			0.687	0.962	1.000	1.000
power	TU Lin.	9/4	2.500	0.100			0.711	0.951	1.000	1.000
	TU Sph.	9/4	3.744	0.100			0.710	0.953	1.000	1.000

Table 4: Simulation Results of Test for Tail-Down with $q = 4$ and $S = 1,000$

The results in Table 2 – 4 show that the size of the test converges to nominal level of 0.05 and the powers for TU models are reasonably high when number of levels l is sufficiently large. The power tends to increase with the growing dependence in the network (larger m). Furthermore, the rejection rates under two TD models are close to those under the TD linear with sill model, which implies fair robustness of this test to violation of m-$(2m+1/q)$-dependence assumption.

Due to the fact that the denominator of T_{td}^m, $\widehat{\sigma}_{11}^m + \widehat{\sigma}_{22}^m - 2\widehat{\sigma}_{12}^m$, is not guaranteed to be positive, it is possible that $T_{td}^m < 0$ when sample size is small. Table 5 – 7 tabulate the proportions of $T_{td}^m < 0$ under each scenario. It is seen that, when the number of levels is sufficiently large (also means large sample size), negative T_{td}^m becomes rare. However, larger m (stronger dependence) requires more levels to achieve this.

Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	1/2	1.000	0.500	0.036	0.022	0.002	0.000	0.000	0.000
TD Sph.	1/2	1.440	0.500	0.035	0.024	0.001	0.000	0.000	0.000
TD Exp.	1/2	0.721	0.500	0.035	0.025	0.002	0.000	0.000	0.000
TU Exp.	1/2	0.721	0.500	0.031	0.016	0.001	0.000	0.000	0.000
TU Lin.	1/2	1.000	0.500	0.037	0.024	0.002	0.000	0.000	0.000
TU Sph.	1/2	1.440	0.500	0.040	0.023	0.000	0.001	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	2/2	1.500	0.333	0.040	0.149	0.062	0.022	0.001	0.000
TD Sph.	2/2	2.212	0.333	0.061	0.061	0.013	0.005	0.000	0.000
TD Exp.	2/2	1.233	0.333	0.044	0.034	0.003	0.002	0.000	0.000
TU Exp.	2/2	1.233	0.333	0.036	0.022	0.001	0.000	0.000	0.000
TU Lin.	2/2	1.500	0.333	0.046	0.067	0.009	0.001	0.000	0.000
TU Sph.	2/2	2.212	0.333	0.034	0.037	0.000	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	3/2	2.000	0.250		0.077	0.023	0.005	0.000	0.000
TD Sph.	3/2	2.972	0.250		0.057	0.007	0.024	0.000	0.000
TD Exp.	3/2	1.738	0.250		0.041	0.006	0.006	0.000	0.000
TU Exp.	3/2	1.738	0.250		0.022	0.001	0.000	0.000	0.000
TU Lin.	3/2	2.000	0.250		0.038	0.007	0.001	0.000	0.000
TU Sph.	3/2	2.972	0.250		0.026	0.003	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	4/2	2.500	0.200		0.081	0.015	0.062	0.014	0.003
TD Sph.	4/2	3.728	0.200		0.086	0.011	0.031	0.005	0.000
TD Exp.	4/2	2.241	0.200		0.037	0.007	0.010	0.000	0.000
TU Exp.	4/2	2.241	0.200		0.024	0.002	0.001	0.000	0.000
TU Lin.	4/2	2.500	0.200		0.032	0.005	0.003	0.000	0.000
TU Sph.	4/2	3.728	0.200		0.024	0.001	0.002	0.000	0.000

Table 5: Proportion of negative T statistic under m-$(2m+1/q)$-dependence models with $q = 2$ and $S = 1,000$

Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
TD Lin.	2/3	1.000	0.333	0.286	0.045	0.008	0.000	0.000	0.000
TD Sph.	2/3	1.474	0.333	0.282	0.046	0.006	0.003	0.000	0.000
TD Exp.	2/3	0.822	0.333	0.236	0.046	0.001	0.000	0.000	0.000
TU Exp.	2/3	0.822	0.333	0.173	0.013	0.001	0.000	0.000	0.000
TU Lin.	2/3	1.000	0.333	0.220	0.023	0.000	0.000	0.000	0.000
TU Sph.	2/3	1.474	0.333	0.192	0.032	0.000	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
TD Lin.	4/3	1.500	0.222		0.103	0.022	0.005	0.000	0.000
TD Sph.	4/3	2.233	0.222		0.074	0.018	0.004	0.000	0.000
TD Exp.	4/3	1.326	0.222		0.070	0.006	0.002	0.000	0.000
TU Exp.	4/3	1.326	0.222		0.021	0.001	0.000	0.000	0.000
TU Lin.	4/3	1.500	0.222		0.022	0.002	0.000	0.000	0.000
TU Sph.	4/3	2.233	0.222		0.022	0.000	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
TD Lin.	5/3	2.000	0.167		0.139	0.039	0.007	0.000	0.000
TD Sph.	5/3	2.988	0.167		0.080	0.008	0.012	0.001	0.000
TD Exp.	5/3	1.828	0.167		0.067	0.014	0.005	0.000	0.000
TU Exp.	5/3	1.828	0.167		0.010	0.000	0.000	0.000	0.000
TU Lin.	5/3	2.000	0.167		0.010	0.001	0.000	0.000	0.000
TU Sph.	5/3	2.988	0.167		0.013	0.000	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels 3	4	5	6	7	8
TD Lin.	7/3	2.500	0.133			0.044	0.032	0.007	0.000
TD Sph.	7/3	3.740	0.133			0.019	0.024	0.001	0.000
TD Exp.	7/3	2.329	0.133			0.011	0.008	0.001	0.000
TU Exp.	7/3	2.329	0.133			0.000	0.000	0.000	0.000
TU Lin.	7/3	2.500	0.133			0.000	0.000	0.000	0.000
TU Sph.	7/3	3.740	0.133			0.000	0.000	0.000	0.000

Table 6: Proportion of negative T statistic under m-$(2m+1/q)$-dependence models with $q = 3$ and $S = 1,000$

Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	3/4	1.000	0.250	0.235	0.052	0.009	0.000	0.001	0.000
TD Sph.	3/4	1.486	0.250	0.264	0.067	0.019	0.001	0.000	0.000
TD Exp.	3/4	0.869	0.250	0.248	0.056	0.029	0.005	0.001	0.000
TU Exp.	3/4	0.869	0.250	0.154	0.013	0.000	0.000	0.000	0.000
TU Lin.	3/4	1.000	0.250	0.142	0.016	0.002	0.000	0.000	0.000
TU Sph.	3/4	1.486	0.250	0.151	0.014	0.001	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	5/4	1.500	0.167		0.138	0.068	0.013	0.002	0.000
TD Sph.	5/4	2.241	0.167		0.102	0.043	0.001	0.002	0.000
TD Exp.	5/4	1.371	0.167		0.087	0.042	0.011	0.005	0.000
TU Exp.	5/4	1.371	0.167		0.007	0.005	0.000	0.000	0.000
TU Lin.	5/4	1.500	0.167		0.012	0.001	0.000	0.000	0.000
TU Sph.	5/4	2.241	0.167		0.016	0.001	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	7/4	2.000	0.125		0.168	0.055	0.014	0.001	0.001
TD Sph.	7/4	2.993	0.125		0.106	0.057	0.008	0.004	0.000
TD Exp.	7/4	1.872	0.125		0.073	0.044	0.009	0.005	0.001
TU Exp.	7/4	1.872	0.125		0.016	0.002	0.000	0.000	0.000
TU Lin.	7/4	2.000	0.125		0.012	0.001	0.000	0.000	0.000
TU Sph.	7/4	2.993	0.125		0.006	0.002	0.000	0.000	0.000
Model	m	θ_r	$\gamma_0(1/q)$	# of Stream Network Levels					
				3	4	5	6	7	8
TD Lin.	9/4	2.500	0.100			0.104	0.034	0.014	0.001
TD Sph.	9/4	3.744	0.100			0.061	0.013	0.004	0.000
TD Exp.	9/4	2.373	0.100			0.057	0.012	0.005	0.000
TU Exp.	9/4	2.373	0.100			0.001	0.000	0.000	0.000
TU Lin.	9/4	2.500	0.100			0.001	0.000	0.000	0.000
TU Sph.	9/4	3.744	0.100			0.003	0.000	0.000	0.000

Table 7: Proportion of negative T statistic under m-$(2m+1/q)$-dependence models with $q = 4$ and $S = 1,000$

3 Extension to Absolutely Summable Covariance Functions

3.1 Methodology of Test for Tail-Down Model

The test constructed in Section 2 shows favorable performance under m-($2m$+$1/q$)-dependence. The next step is to extend this test for tail-down model to stream network data with absolutely summable covariance functions.

Definition 3.1.1. *Let $\{Z_{ij} : i \in A, j = 1, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with the network and covariance functions as specified in Section 2.1. Then, $\{Z_{ij}\}$ is said to have absolutely summable covariance functions if*

$$\sum_{i=0}^{\infty} \Big| C_{fc}(i/q) \Big| < \infty, \quad \sum_{i=0}^{\infty} \sum_{j=0}^{i} 2^{j/2} \Big| C_{fu}(1/2q + j/q, 1/2q + i/q) \Big| < \infty.$$

In the same spirit of T_{td}^m, a similar test statistic can be constructed and used if $\sqrt{|A|}\left(\underline{\widehat{\gamma}} - \underline{\gamma}\right)$ is asymptotically normal, and consistent estimators of limiting variances and covariances exist. In order to distinguish quantities under m-($2m$+$1/q$)-dependent model from those under absolutely summable covariance functions, from this section, superscript m is used for quantities under m-($2m$+$1/q$)-dependent model. More notations are listed in Table 8.

	m-($2m$+$1/q$)-Dependent	Absolutely Summable
Covariance Functions	C_{fc}^m, C_{fu}^m	C_{fc}, C_{fu}
Theoretical Semivariograms	$\gamma_0^m, \gamma_1^m, \gamma_c^m, \gamma_u^m$	$\gamma_0, \gamma_1, \gamma_c, \gamma_u$
Sample Semivariograms	$\widehat{\gamma}_0^m, \widehat{\gamma}_1^m, \widehat{\gamma}_c^m, \widehat{\gamma}_u^m$	$\widehat{\gamma}_0, \widehat{\gamma}_1, \widehat{\gamma}_c, \widehat{\gamma}_u$
Asymptotic Variance Matrix	$\mathbf{\Sigma}^m$	$\mathbf{\Sigma}$
Asymptotic Variances/Covariances	$\sigma_{11}^m, \sigma_{12}^m, \sigma_{22}^m$	$\sigma_{11}, \sigma_{12}, \sigma_{22}$

Table 8: Notations of quantities under two covariance model settings

The objective of subsequent subsections is to show that the limiting distribution of $\sqrt{|A|}\left(\underline{\widehat{\gamma}} - \underline{\gamma}\right)$ is multivariate normal with mean $\underline{0}$ and some covariance $\mathbf{\Sigma}$. Inspired by Lu and Zimmerman (2001), Proposition 6.3.9 of Brockwell and Davis (1991) can be used to achieve this objective.

Proposition 6.3.9 (Brockwell and Davis 1991)
Let X_n, $n = 1, 2, ...$ and Y_{nj}, $j = 1, 2, ...; n = 1, 2, ...$, be random k-vectors such that

(i) $Y_{nj} \xrightarrow{d} Y_j$ as $n \to \infty$ for each $j = 1, 2, ...$,

(ii) $Y_j \xrightarrow{d} Y$ as $j \to \infty$, and

(iii) $\lim_{j\to\infty} \limsup_{n\to\infty} P\left(\left|X_n - Y_{nj}\right| > \epsilon\right) = 0$ for every $\epsilon > 0$.

Then

$$X_n \xrightarrow{d} Y \text{ as } n \to \infty.$$

Hence, the asymptotic normality of random vector $\sqrt{|A|}\left(\widehat{\underline{\gamma}} - \underline{\gamma}\right)$ can be proved given three conditions (c1)–(c3) as follows:

(c1). For any m-$(2m{+}1/q)$-dependent $\{Z_{ij}^m\}$, $\sqrt{|A|}\left(\widehat{\underline{\gamma}}^m - \underline{\gamma}^m\right) \xrightarrow{d} N(\underline{0}, \mathbf{\Sigma}^m)$ as $l_{sn} \to \infty$,

(c2). $\mathbf{\Sigma}^m \to \mathbf{\Sigma}$ for some $\mathbf{\Sigma}$ as $m \to \infty$,

(c3). $\lim_{m\to\infty} \limsup_{l_{sn}\to\infty} \mathrm{P}\left(\sqrt{|A|}\left|\left(\widehat{\underline{\gamma}} - \underline{\gamma}\right) - \left(\widehat{\underline{\gamma}}^m - \underline{\gamma}^m\right)\right| > \epsilon\right) = 0$ for every $\epsilon > 0$.

The following subsections are arranged to first check the conditions (c1)–(c3), then prove the asymptotic normality of $\sqrt{|A|}\left(\widehat{\underline{\gamma}} - \underline{\gamma}\right)$ in detail. Section 3.2 checks condition (c1), Section 3.3–3.4 check condition (c2), Section 3.5 checks condition (c3) and finalizes the proof of asymptotic normality. Additionally, Section 3.6 presents consistent estimators of the elements of $\mathbf{\Sigma}$ and proves the consistency.

3.2 Asymptotic Normality under m-$(2m{+}1/q)$-Dependence

The following theorem is simply a modified version of Theorem 2.3.1 with elements as infinite sums rather than finite ones. Additional $*$ superscript is added to distinguish the asymptotic variances and covariances as infinite sums from those in Section 2.2.

Theorem 3.2.1. *Let $\{Z_{ij}^m : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with network and covariance function specified as in Section 2.1. Then, under m-$(2m + 1/q)$-dependence, the random vector*

$$\sqrt{|A|}\left(\widehat{\underline{\gamma}}^m - \underline{\gamma}^m\right) \xrightarrow{d} N\left(\underline{0}, \mathbf{\Sigma}^{m*}\right) \;\; \textit{as } l_{sn} \to \infty,$$

with the elements of $\boldsymbol{\Sigma}^{m*}$ *as*

$$
\begin{aligned}
\sigma_{11}^{m*} =& \frac{1}{2(q-1)}\left[2\gamma_0^m(1/q)\right]^2 + \frac{q-2}{(q-1)^2}\left[2\gamma_0^m(1/q)-\gamma_c^m(2/q)\right]^2 + \frac{(q-3)\vee 0}{(q-1)^2}\left[\gamma_0^m(1/q)-2\gamma_c^m(2/q)+\gamma_c^m(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=3}^{(qm+1)\wedge(q-2)}(q-k-1)\Big\{\gamma_c^m[(k-1)/q]-2\gamma_c^m(k/q)+\gamma_c^m[(k+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\left[\gamma_1^m(1/q)-2\gamma_c^m(2/q)+\gamma_c^m(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\gamma_c^m[(l-1)/q]-2\gamma_c^m(l/q)+\gamma_c^m[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=2}^{\infty}\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\gamma_c^m[(l-1)/q]-2\gamma_c^m(l/q)+\gamma_c^m[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\infty}2^{k-1}\sum_{u=qk}^{q(k+1)-2}\Big\{\gamma_u^m[(2u+1)/2q,(2u+1)/2q]-2\gamma_u^m[(2u+1)/2q,(2u+3)/2q] \\
&\qquad\qquad + \gamma_u^m[(2u+3)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\infty}2^{k}\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\gamma_u^m[(2v+1)/2q,(2u+1)/2q]-\gamma_u^m[(2v+3)/2q,(2u+1)/2q] \\
&\qquad\qquad - \gamma_u^m[(2v+1)/2q,(2u+3)/2q]+\gamma_u^m[(2v+3)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}2^{l}\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\gamma_u^m[(2v+1)/2q,(2u+1)/2q]-\gamma_u^m[(2v+3)/2q,(2u+1)/2q] \\
&\qquad\qquad - \gamma_u^m[(2v+1)/2q,(2u+3)/2q]+\gamma_u^m[(2v+3)/2q,(2u+3)/2q]\Big\}^2, \\
\sigma_{22}^{m*} =& \frac{1}{2}\left[2\gamma_1^m(1/q)\right]^2 + \frac{1}{2}\left[2\gamma_1^m(1/q)-\gamma_u^m(1/2q,1/2q)\right]^2 + \mathrm{I}\{q=2\}\cdot\mathrm{I}\Big\{\lfloor m+1/q\rfloor\geq 1\Big\}\cdot\left[\gamma_0^m(1/q)-2\gamma_c^m(2/q)+\gamma_c^m(3/q)\right]^2 \\
&+ \mathrm{I}\{q\geq 3\}\cdot\mathrm{I}\Big\{\lfloor m+1/q\rfloor\geq 1\Big\}\cdot\left[\gamma_c^m(1-1/q)-2\gamma_c^m(1)+\gamma_c^m(1+1/q)\right]^2 \\
&+ \sum_{k=2}^{\infty}\left[\gamma_c^m(k-1/q)-2\gamma_c^m(k)+\gamma_c^m(k+1/q)\right]^2 \\
&+ \sum_{k=1}^{\infty}\left[\gamma_c^m(k)-\gamma_c^m(k+1/q)-\gamma_u^m(1/2q,k-1/2q)+\gamma_u^m(1/2q,k+1/2q)\right]^2 \\
&+ \sum_{k=1}^{\infty}2^{k-1}\left[\gamma_u^m(k-1/2q,k-1/2q)-2\gamma_u^m(k-1/2q,k+1/2q)+\gamma_u^m(k+1/2q,k+1/2q)\right]^2 \\
&+ \sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^{l}\big[\gamma_u^m(l-1/2q,k-1/2q)-\gamma_u^m(l+1/2q,k-1/2q) \\
&\qquad\qquad - \gamma_u^m(l-1/2q,k+1/2q)+\gamma_u^m(l+1/2q,k+1/2q)\big]^2,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{12}^{m*} =& \frac{1}{q-1}\left[\gamma_0^m(1/q)+\gamma_1^m(1/q)-\gamma_c^m(2/q)\right]^2 + \mathrm{I}\{q\geq 3\}\cdot\frac{1}{q-1}\left[\gamma_0^m(1/q)-2\gamma_c^m(2/q)+\gamma_c^m(3/q)\right]^2 \\
&+\frac{1}{q-1}\sum_{l=3}^{q-1}\Big\{\gamma_c^m\left[(l-1)/q\right]-2\gamma_c^m(l/q)+\gamma_c^m\left[(l+1)/q\right]\Big\}^2 \\
&+\frac{1}{q-1}\sum_{k=1}^{\infty}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c^m\left[(l-1)/q\right]-2\gamma_c^m(l/q)+\gamma_c^m\left[(l+1)/q\right]\Big\}^2 \\
&+\frac{1}{2(q-1)}\left[\gamma_1^m(1/q)-\gamma_c^m(2/q)-\gamma_u^m(1/2q,1/2q)+\gamma_u^m(1/2q,3/2q)\right]^2 \\
&+\mathrm{I}\{q\geq 3\}\cdot\frac{1}{2(q-1)}\left[\gamma_c^m(2/q)-\gamma_c^m(3/q)-\gamma_u^m(1/2q,3/2q)+\gamma_u^m(1/2q,5/2q)\right]^2 \\
&+\frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\gamma_c^m[(l+1)/q]-\gamma_c^m[(l+2)/q]-\gamma_u^m[1/2q,(2l+1)/2q]+\gamma_u^m[1/2q,(2l+3)/2q]\Big\}^2 \\
&+\frac{1}{2(q-1)}\sum_{k=1}^{\infty}\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c^m[(l+1)/q]-\gamma_c^m[(l+2)/q]-\gamma_u^m[1/2q,(2l+1)/2q]+\gamma_u^m[1/2q,(2l+3)/2q]\Big\}^2 \\
&+\frac{1}{q-1}\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^{l-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u^m[(2lq-1)/2q,(2u+1)/2q]-\gamma_u^m[(2lq+1)/2q,(2u+1)/2q] \\
&\qquad -\gamma_u^m[(2lq-1)/2q,(2u+3)/2q]+\gamma_u^m[(2lq+1)/2q,(2u+3)/2q]\Big\}^2 \\
&+\frac{1}{q-1}\sum_{l=1}^{\infty}\sum_{k=1}^{l}2^{k-2}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u^m[(2u+1)/2q,(2lq-1)/2q]-\gamma_u^m[(2u+1)/2q,(2lq+1)/2q] \\
&\qquad -\gamma_u^m[(2u+3)/2q,(2lq-1)/2q]+\gamma_u^m[(2u+3)/2q,(2lq+1)/2q]\Big\}^2.
\end{aligned}
$$

Proof. This proof is greatly based on theorems in Section 2.2 and 2.3, and it would suffice to show that three unique elements of $\mathbf{\Sigma}^{m*}$ are equal to those in Theorem 2.2.1 – 2.2.3. σ_{11}^{m*} equals to σ_{22}^{m} in Theorem 2.2.2 is elaborated in this proof, and σ_{11}^{m*} and σ_{22}^{m*} parts can be verified in similar fashion. By Theorem 2.2.2,

$$
\begin{aligned}
&\sigma_{22}^{m*}-\sigma_{22}^{m} \\
=&\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\gamma_c^m(k-1/q)-2\gamma_c^m(k)+\gamma_c^m(k+1/q)\right]^2 \\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\gamma_c^m(k)-\gamma_c^m(k+1/q)-\gamma_u^m(1/2q,k-1/2q)+\gamma_u^m(1/2q,k+1/2q)\right]^2 \\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}2^{k-1}\left[\gamma_u^m(k-1/2q,k-1/2q)-2\gamma_u^m(k-1/2q,k+1/2q)+\gamma_u^m(k+1/2q,k+1/2q)\right]^2 \\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\sum_{l=1}^{k-1}2^{l}\big[\gamma_u^m(l-1/2q,k-1/2q)-\gamma_u^m(l+1/2q,k-1/2q) \\
&\qquad -\gamma_u^m(l-1/2q,k+1/2q)+\gamma_u^m(l+1/2q,k+1/2q)\big]^2. \qquad \text{(a.3.1)}
\end{aligned}
$$

By the definition of m-$(2m{+}1/q)$-dependence, when $k-1/q\geq\lfloor m+1/q\rfloor+1-1/q\geq m+1/q$, $C_{fc}^m(k-1/q)=0$ and $\gamma_c^m(k-1/q)=C_{fc}^m(0)-C_{fc}^m(k-1/q)=\sigma^2$. Moreover, $\gamma_c^m(k)=\gamma_c^m(k+0.5)=\sigma^2$, then

$$\gamma_c^m(k - 1/q) - 2\gamma_c^m(k) + \gamma_c^m(k + 1/q) = 0$$

for $k = \lfloor m+1/q \rfloor + 1, \lfloor m+1/q \rfloor + 2, \ldots$ Thus the first term of (a.3.1) is zero. Similarly, by the definition of m-$(2m+1/q)$-dependence, when $k - 1/2q \geq \lfloor m+1/q \rfloor + 1 - 1/2q > m + 1/q$, $C_{fu}^m(\cdot, k - 1/2q) = 0$ and $\gamma_u^m(k - 1/2q, k - 1/2q) = C_{fu}^m(0, 0) - C_{fu}^m(k - 1/2q, k - 1/2q) = \sigma^2$. Moreover, $\gamma_u^m(k - 1/2q, k + 1/2q) = \gamma_u^m(k + 1/2q, k + 1/2q) = \sigma^2$, then

$$\gamma_u^m(k - 1/2q, k - 1/2q) - 2\gamma_u^m(k - 1/2q, k + 1/2q) + \gamma_u^m(k + 1/2q, k + 1/2q) = 0$$

for $k = \lfloor m + 1/q \rfloor + 1, \lfloor m + 1/q \rfloor + 2, \ldots$ Thus the third term of (a.3.1) is zero. A similar process would show that other terms of (a.3.1) are also zero. Hence, $\sigma_{22}^{m*} - \sigma_{22}^m = 0$, or $\sigma_{22}^{m*} = \sigma_{22}^m$. Recall that $\sigma_{11}^{m*} = \sigma_{11}^m$ and $\sigma_{12}^{m*} = \sigma_{12}^m$, where σ_{11}^m and σ_{12}^m are from Theorem 2.2.1 and Theorem 2.2.3, can be proved in similar fashion. This completes the proof. $\square$

3.3 m-$(2m+1/q)$-Dependent Gaussian Random Field Derived from Truncated Moving Average Function on Stream Network

In order to prove the asymptotic normality of sample semivariograms on a grid with absolutely summable covariance function, Lu and Zimmerman (2001) constructed an (m_x, m_y)-dependent Gaussian random field that converges to the base Gaussian random field. Assuming that $\{Z(\underline{s})\}$ is the base Gaussian random field on a 2-dimensional grid, a new (m_x, m_y)-dependent Gaussian random field $\{Z^m(\underline{s})\}$ can be specified such that the $\mathrm{E}[Z^m(\underline{s})] = \mathrm{E}[Z(\underline{s})]$ and covariances are the covariances of $\{Z(\underline{s})\}$ truncated at maximum lags of m_x and m_y in two dimensions.

A naive attempt to apply this strategy directly to stream network is improper because truncated covariance functions based on valid covariance functions is not guaranteed to be valid on a stream network. That means truncated covariance functions can yield a covariance matrix that is not positive-definite.

Assume that $\{Z_{ij} : i \in A, j = 1, 2, \ldots, q\}$ is a base Gaussian random field on the stream network as specified in Section 2.1, and the covariance functions can be derived from integrating a non-negative moving average function, $g(x)$, with respect to the white noise variables along the stream (Ver Hoef and Peterson 2010). Moreover, $g(x)$ is unilateral, which means $g(x)$ is zero for any $x < 0$ and decreasing when $x \geq 0$. $\mathrm{E}(Z_{ij}) = \mu$ and the covariance functions are

$$C_{td}(h, a, b) = \begin{cases} C_{fc}(h) = \int_{-\infty}^{-h} g(-x)g(-x-h)\, dx & \text{flow connected,} \\ C_{fu}(a, b) = \int_{-\infty}^{-b} g(-x)g(-x-(b-a))\, dx & \text{flow unconnected.} \end{cases}$$

Instead of directly truncating the covariance functions, the new m-$(2m+1/q)$-dependent Gaussian random field, $\{Z_{ij}^m : i \in A, j = 1, 2, ..., q\}$, are based on a truncated moving average function, $g^m(x)$. More specifically,

$$g^m(x) = g(x) \cdot \mathrm{I}\{0 \leq x \leq m + 1/q\}. \tag{5}$$

Then $\mathrm{E}(Z_{ij}^m) = \mu$ and the covariance functions of $\{Z_{ij}^m\}$ are

$$C_{td}^m(h, a, b) = \begin{cases} C_{fc}^m(h) = \int_{-\infty}^{-h} g^m(-x)g^m(-x-h)\, dx & \text{flow connected,} \\ C_{fu}^m(a, b) = \int_{-\infty}^{-b} g^m(-x)g^m(-x-(b-a))\, dx & \text{flow unconnected.} \end{cases}$$

The following theorem establishes the pointwise convergence of $C_{fc}^m(h)$ and $C_{fu}^m(a,b)$ to $C_{fc}(h)$ and $C_{fu}(a,b)$, respectively, as $m \to \infty$.

Theorem 3.3.1. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with stream network and covariance functions specified as in Section 2.1. Assume that $g(\cdot)$ is the moving average function integrated over a white-noise random process along the stream network to construct covariance functions. For arbitrary $m \in \{x/q : x \in \mathbb{N}_+\}$, let $\{Z_{ij}^m : i \in A, j = 1, 2\}$ be another Gaussian random field on the same steam network. Let $\mathrm{E}(Z_{ij}^m) = \mathrm{E}(Z_{ij})$ and let the covariance functions of $\{Z_{ij}^m\}$ be derived from a truncated moving average function $g^m(\cdot)$ such that*

$$g^m(x) = g(x) \cdot \mathrm{I}\{0 \leq x \leq m + 1/q\}.$$

Define the covariance functions as (assume $0 \le a \le b$)

$$C^m_{td}(h,a,b) = \begin{cases} C^m_{fc}(h) & \textit{flow connected,} \\ C^m_{fu}(a,b) & \textit{flow unconnected.} \end{cases}$$

Then for any fixed (h,a,b) such that $h = a+b \ge 0$ and $b \ge a \ge 0$, as $m \to \infty$, $C^m_{fc}(h) \to C_{fc}(h)$ and $C^m_{fu}(a,b) \to C_{fu}(a,b)$.

Proof. See Appendix A.3. □

3.4 Convergence of $\mathbf{\Sigma}^m$ to $\mathbf{\Sigma}$

Theorem 3.4.1. *Let $\{Z_{ij} : i \in A, j = 1,2,...,q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with stream network and covariance functions specified as in Section 2.1. Assume that the covariance functions of $\{Z_{ij}\}$ are absolutely summable. For arbitrary $m \in \{x/q : x \in \mathbb{N}_+\}$, let $\{Z^m_{ij} : i \in A, j = 1,2,...,q\}$ be another Gaussian random field constructed as in Theorem 3.3.1. Let $\mathbf{\Sigma}^m$ be the asymptotic covariance matrix of $\sqrt{|A|}\left(\underline{\widehat{\gamma}}^m - \underline{\gamma}^m\right)$ as $l_{sn} \to \infty$ as specified in Theorem 2.3.1. Then $\mathbf{\Sigma}^m \to \mathbf{\Sigma}$ as $m \to \infty$ for some bounded matrix $\mathbf{\Sigma}$, whose elements are*

$$\begin{aligned}
\sigma_{11} =& \frac{1}{2(q-1)}\left[2\gamma_0(1/q)\right]^2 + \frac{q-2}{(q-1)^2}\left[2\gamma_0(1/q) - \gamma_c(2/q)\right]^2 + \frac{(q-3)\vee 0}{(q-1)^2}\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=3}^{q-2}(q-k-1)\Big\{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\left[\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ \frac{1}{(q-1)^2}\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=2}^{\infty}\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\infty}2^{k-1}\sum_{u=qk}^{q(k+1)-2}\Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q] - 2\gamma_u[(2u+1)/2q,(2u+3)/2q] \\
&\qquad + \gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{(q-1)^2}\sum_{k=0}^{\infty}2^{k}\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q] - \gamma_u[(2v+3)/2q,(2u+1)/2q] \\
&\qquad - \gamma_u[(2v+1)/2q,(2u+3)/2q] + \gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2
\end{aligned}$$

$$+\frac{1}{(q-1)^2}\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}2^l\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]$$

$$-\gamma_u[(2v+1)/2q,(2u+3)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2,$$

$$\sigma_{22}=\frac{1}{2}\left[2\gamma_1(1/q)\right]^2+\frac{1}{2}\left[2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)\right]^2+\mathrm{I}\{q=2\}\cdot\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2$$

$$+\mathrm{I}\{q\geq 3\}\cdot\left[\gamma_c(1-1/q)-2\gamma_c(1)+\gamma_c(1+1/q)\right]^2$$

$$+\sum_{k=2}^{\infty}\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2$$

$$+\sum_{k=1}^{\infty}\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2$$

$$+\sum_{k=1}^{\infty}2^{k-1}\left[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)+\gamma_u(k+1/2q,k+1/2q)\right]^2$$

$$+\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^l\Big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)$$

$$-\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\Big]^2,$$

$$\sigma_{12}=\frac{1}{q-1}\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2+\mathrm{I}\{q\geq 3\}\cdot\frac{1}{q-1}\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2$$

$$+\frac{1}{q-1}\sum_{l=3}^{q-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2$$

$$+\frac{1}{q-1}\sum_{k=1}^{\infty}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2$$

$$+\frac{1}{2(q-1)}\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2$$

$$+\mathrm{I}\{q\geq 3\}\cdot\frac{1}{2(q-1)}\left[\gamma_c(2/q)-\gamma_c(3/q)-\gamma_u(1/2q,3/2q)+\gamma_u(1/2q,5/2q)\right]^2$$

$$+\frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2$$

$$+\frac{1}{2(q-1)}\sum_{k=1}^{\infty}\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2$$

$$+\frac{1}{q-1}\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^{l-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2lq-1)/2q,(2u+1)/2q]-\gamma_u[(2lq+1)/2q,(2u+1)/2q]$$

$$-\gamma_u[(2lq-1)/2q,(2u+3)/2q]+\gamma_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2$$

$$+\frac{1}{q-1}\sum_{l=1}^{\infty}\sum_{k=1}^{l}2^{k-2}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2u+1)/2q,(2lq-1)/2q]-\gamma_u[(2u+1)/2q,(2lq+1)/2q]$$

$$-\gamma_u[(2u+3)/2q,(2lq-1)/2q]+\gamma_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2.$$

Proof. See Appendix A.4. □

3.5 Asymptotic Normality under Absolutely Summable Covariance Functions

Theorem 3.5.1. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with stream network and covariance functions specified as in Section 2.1. Assume that the covariance functions of $\{Z_{ij}\}$ are absolutely summable. Then $\sqrt{|A|}\left(\underline{\widehat{\gamma}} - \underline{\gamma}\right) \xrightarrow{d} N\left(\underline{0}, \mathbf{\Sigma}\right)$ as $l_{sn} \to \infty$, and the elements of $\mathbf{\Sigma}$ are specified as Theorem 3.4.1.*

Proof. As aforementioned in Section 3.1, by Proposition 6.3.9 of Brockwell and Davis (1991) and with $\{Z_{ij}^m\}$ constructed as in Theorem 3.3.1, Conditions (c1)–(c3) implies that $\sqrt{|A|}\left(\underline{\widehat{\gamma}} - \underline{\gamma}\right) \xrightarrow{d} N\left(\underline{0}, \mathbf{\Sigma}\right)$ as $l_{sn} \to \infty$. Conditions (c1) and (c2) are proved by Theorem 2.3.1 and Theorem 3.3.1. Hence the proof of condition (c3) completes the proof of asymptotic normality.

For any $\epsilon > 0$,

$$\begin{aligned}
&\mathrm{P}\left(\sqrt{|A|}\Big|\left(\underline{\widehat{\gamma}} - \underline{\gamma}\right) - \left(\underline{\widehat{\gamma}}^m - \underline{\gamma}^m\right)\Big| > \epsilon\right)\\
=&\mathrm{P}\Bigg(\Big[|A|\left[(\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q))\right]^2\\
&\qquad + |A|\left[(\widehat{\gamma}_1(1/q) - \gamma_1(1/q)) - (\widehat{\gamma}_1^m(1/q) - \gamma_1^m(1/q))\right]^2\Big]^{1/2} > \epsilon\Bigg)\\
\le&\mathrm{P}\Bigg(\left\{\sqrt{|A|}\Big|(\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q))\Big| > \epsilon/2\right\}\cup\\
&\qquad \left\{\sqrt{|A|}\Big|(\widehat{\gamma}_1(1/q) - \gamma_1(1/q)) - (\widehat{\gamma}_1^m(1/q) - \gamma_1^m(1/q))\Big| > \epsilon/2\right\}\Bigg)\\
\le&\mathrm{P}\Bigg(\sqrt{|A|}\Big|(\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q))\Big| > \epsilon/2\Bigg)\\
&+\mathrm{P}\Bigg(\sqrt{|A|}\Big|(\widehat{\gamma}_1(1/q) - \gamma_1(1/q)) - (\widehat{\gamma}_1^m(1/q) - \gamma_1^m(1/q))\Big| > \epsilon/2\Bigg).
\end{aligned} \tag{6}$$

By Chebyshev's inequality,

$$\begin{aligned}
&\mathrm{P}\Bigg(\sqrt{|A|}\Big|(\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q))\Big| > \epsilon/2\Bigg)\\
\le&\frac{4|A|\mathrm{Var}\left[(\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q))\right]}{\epsilon^2} = \frac{4|A|\mathrm{Var}\left(\widehat{\gamma}_0(1/q) - \widehat{\gamma}_0^m(1/q)\right)}{\epsilon^2}.
\end{aligned}$$

By the fact that $\widehat{\gamma}_0(1/q) = \widehat{\gamma}_0^m(1/q) = \frac{1}{2|A|(q-1)} \sum_{i=1}^{|A|} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2$,

$$\lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\sqrt{|A|} \Big| (\widehat{\gamma}_0(1/q) - \gamma_0(1/q)) - (\widehat{\gamma}_0^m(1/q) - \gamma_0^m(1/q)) \Big| > \epsilon/2 \right) = 0.$$

Similarly,

$$\lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\sqrt{|A|} \Big| (\widehat{\gamma}_1(1/q) - \gamma_1(1/q)) - (\widehat{\gamma}_1^m(1/q) - \gamma_1^m(1/q)) \Big| > \epsilon/2 \right) = 0.$$

Then by (6),

$$\begin{aligned} &\lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\sqrt{|A|} \Big| (\underline{\widehat{\gamma}} - \underline{\gamma}) - (\underline{\widehat{\gamma}}^m - \underline{\gamma}^m) \Big| > \epsilon \right) \\ =& \lim_{m\to\infty} \limsup_{l_{sn}\to\infty} \mathrm{P}\left(\sqrt{|A|} \Big| (\underline{\widehat{\gamma}} - \underline{\gamma}) - (\underline{\widehat{\gamma}}^m - \underline{\gamma}^m) \Big| > \epsilon \right) = 0. \end{aligned}$$

Hence condition (c3) is satisfied. This completes the proof. □

3.6 Consistency of $\widehat{\boldsymbol{\Sigma}}$ under Absolutely Summable Covariance Functions

Theorem 3.6.1. *Let $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ be a second-order stationary Gaussian random field on a regular rooted binary tree stream network, with stream network and covariance functions specified as in Section 3.1. Assume that the covariance functions of $\{Z_{ij}\}$ are absolutely summable. Let $\widehat{\boldsymbol{\Sigma}}$ be a 2×2 matrix with elements*

$$\begin{aligned} \widehat{\sigma}_{11} =& \frac{1}{2(q-1)} [2\widehat{\gamma}_0(1/q)]^2 + \frac{q-2}{(q-1)^2} [2\widehat{\gamma}_0(1/q) - \widehat{\gamma}_c(2/q)]^2 + \frac{(q-3)\vee 0}{(q-1)^2} [\widehat{\gamma}_0(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)]^2 \\ &+ \frac{1}{(q-1)^2} \sum_{k=3}^{q-2} (q-k-1) \Big\{ \widehat{\gamma}_c[(k-1)/q] - 2\widehat{\gamma}_c(k/q) + \widehat{\gamma}_c[(k+1)/q] \Big\}^2 \\ &+ \frac{1}{(q-1)^2} [\widehat{\gamma}_1(1/q) - 2\widehat{\gamma}_c(2/q) + \widehat{\gamma}_c(3/q)]^2 \\ &+ \frac{1}{(q-1)^2} \sum_{l=3}^{2q-2} [q-1-(l-q)\vee(q-l)] \Big\{ \widehat{\gamma}_c[(l-1)/q] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c[(l+1)/q] \Big\}^2 \\ &+ \frac{1}{(q-1)^2} \sum_{k=2}^{\infty} \sum_{l=q(k-1)+2}^{q(k+1)-2} [q-1-(l-qk)\vee(qk-l)] \Big\{ \widehat{\gamma}_c[(l-1)/q] - 2\widehat{\gamma}_c(l/q) + \widehat{\gamma}_c[(l+1)/q] \Big\}^2 \\ &+ \frac{1}{(q-1)^2} \sum_{k=0}^{\infty} 2^{k-1} \sum_{u=qk}^{q(k+1)-2} \Big\{ \widehat{\gamma}_u[(2u+1)/2q, (2u+1)/2q] - 2\widehat{\gamma}_u[(2u+1)/2q, (2u+3)/2q] \\ &\qquad + \widehat{\gamma}_u[(2u+3)/2q, (2u+3)/2q] \Big\}^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(q-1)^2}\sum_{k=0}^{\infty}2^k\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\widehat{\gamma}_u[(2v+1)/2q,(2u+1)/2q]-\widehat{\gamma}_u[(2v+3)/2q,(2u+1)/2q]\\
&\qquad-\widehat{\gamma}_u[(2v+1)/2q,(2u+3)/2q]+\widehat{\gamma}_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2\\
&+\frac{1}{(q-1)^2}\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}2^l\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\widehat{\gamma}_u[(2v+1)/2q,(2u+1)/2q]-\widehat{\gamma}_u[(2v+3)/2q,(2u+1)/2q]\\
&\qquad-\widehat{\gamma}_u[(2v+1)/2q,(2u+3)/2q]+\widehat{\gamma}_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\sigma}_{22}=&\frac{1}{2}\left[2\widehat{\gamma}_1(1/q)\right]^2+\frac{1}{2}\left[2\widehat{\gamma}_1(1/q)-\widehat{\gamma}_u(1/2q,1/2q)\right]^2+\mathrm{I}\{q=2\}\cdot\left[\widehat{\gamma}_0(1/q)-2\widehat{\gamma}_c(2/q)+\widehat{\gamma}_c(3/q)\right]^2\\
&+\mathrm{I}\{q\geq 3\}\cdot\left[\widehat{\gamma}_c(1-1/q)-2\widehat{\gamma}_c(1)+\widehat{\gamma}_c(1+1/q)\right]^2\\
&+\sum_{k=2}^{\infty}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]^2\\
&+\sum_{k=1}^{\infty}\left[\widehat{\gamma}_c(k)-\widehat{\gamma}_c(k+1/q)-\widehat{\gamma}_u(1/2q,k-1/2q)+\widehat{\gamma}_u(1/2q,k+1/2q)\right]^2\\
&+\sum_{k=1}^{\infty}2^{k-1}\left[\widehat{\gamma}_u(k-1/2q,k-1/2q)-2\widehat{\gamma}_u(k-1/2q,k+1/2q)+\widehat{\gamma}_u(k+1/2q,k+1/2q)\right]^2\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^l\left[\widehat{\gamma}_u(l-1/2q,k-1/2q)-\widehat{\gamma}_u(l+1/2q,k-1/2q)\right.\\
&\qquad\left.-\widehat{\gamma}_u(l-1/2q,k+1/2q)+\widehat{\gamma}_u(l+1/2q,k+1/2q)\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\sigma}_{12}=&\frac{1}{q-1}\left[\widehat{\gamma}_0(1/q)+\widehat{\gamma}_1(1/q)-\widehat{\gamma}_c(2/q)\right]^2+\mathrm{I}\{q\geq 3\}\cdot\frac{1}{q-1}\left[\widehat{\gamma}_0(1/q)-2\widehat{\gamma}_c(2/q)+\widehat{\gamma}_c(3/q)\right]^2\\
&+\frac{1}{q-1}\sum_{l=3}^{q-1}\Big\{\widehat{\gamma}_c\left[(l-1)/q\right]-2\widehat{\gamma}_c(l/q)+\widehat{\gamma}_c\left[(l+1)/q\right]\Big\}^2\\
&+\frac{1}{q-1}\sum_{k=1}^{\infty}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\widehat{\gamma}_c\left[(l-1)/q\right]-2\widehat{\gamma}_c(l/q)+\widehat{\gamma}_c\left[(l+1)/q\right]\Big\}^2\\
&+\frac{1}{2(q-1)}\left[\widehat{\gamma}_1(1/q)-\widehat{\gamma}_c(2/q)-\widehat{\gamma}_u(1/2q,1/2q)+\widehat{\gamma}_u(1/2q,3/2q)\right]^2\\
&+\mathrm{I}\{q\geq 3\}\cdot\frac{1}{2(q-1)}\left[\widehat{\gamma}_c(2/q)-\widehat{\gamma}_c(3/q)-\widehat{\gamma}_u(1/2q,3/2q)+\widehat{\gamma}_u(1/2q,5/2q)\right]^2\\
&+\frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\widehat{\gamma}_c[(l+1)/q]-\widehat{\gamma}_c[(l+2)/q]-\widehat{\gamma}_u[1/2q,(2l+1)/2q]+\widehat{\gamma}_u[1/2q,(2l+3)/2q]\Big\}^2\\
&+\frac{1}{2(q-1)}\sum_{k=1}^{\infty}\sum_{l=qk}^{q(k+1)-2}\Big\{\widehat{\gamma}_c[(l+1)/q]-\widehat{\gamma}_c[(l+2)/q]-\widehat{\gamma}_u[1/2q,(2l+1)/2q]+\widehat{\gamma}_u[1/2q,(2l+3)/2q]\Big\}^2\\
&+\frac{1}{q-1}\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^{l-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\widehat{\gamma}_u[(2lq-1)/2q,(2u+1)/2q]-\widehat{\gamma}_u[(2lq+1)/2q,(2u+1)/2q]\\
&\qquad-\widehat{\gamma}_u[(2lq-1)/2q,(2u+3)/2q]+\widehat{\gamma}_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2\\
&+\frac{1}{q-1}\sum_{l=1}^{\infty}\sum_{k=1}^{l}2^{k-2}\sum_{u=q(k-1)}^{qk-2}\Big\{\widehat{\gamma}_u[(2u+1)/2q,(2lq-1)/2q]-\widehat{\gamma}_u[(2u+1)/2q,(2lq+1)/2q]\\
&\qquad-\widehat{\gamma}_u[(2u+3)/2q,(2lq-1)/2q]+\widehat{\gamma}_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2,
\end{aligned}
$$

where $\widehat{\gamma}_c(h)$ *and* $\widehat{\gamma}_u(a,b)$ *are regular method of moment estimators of FCSD and FUDJ semivariograms with different lags or distances-to-common-junction. If* $\sup_{l_{sn}} \mathrm{E}\left(\widehat{\sigma}_{11}\right), \sup_{l_{sn}} \mathrm{E}\left(\widehat{\sigma}_{12}\right), \sup_{l_{sn}} \mathrm{E}\left(\widehat{\sigma}_{22}\right) < \infty$*, then* $\widehat{\sigma}_{11} \xrightarrow{p} \sigma_{11}$*,* $\widehat{\sigma}_{12} \xrightarrow{p} \sigma_{12}$ *and* $\widehat{\sigma}_{22} \xrightarrow{p} \sigma_{22}$ *as* $l_{sn} \to \infty$*, where* σ_{11}*,* σ_{12} *and* σ_{22} *are as specified in Theorem 3.5.1.*

Proof. See Appendix A.5. □

Let $\widehat{\sigma}_{ij}^{q}$ be the estimator of σ_{ij} that has the same expression as $\widehat{\sigma}_{ij}^{m}$ (specified in Theorem 2.4.1) but with m replaced by $\widehat{m}$ such that semivariograms can be well estimated. The proof above implies that $\widehat{\sigma}_{11}^{\widehat{m}} \xrightarrow{p} \sigma_{11}$, $\widehat{\sigma}_{12}^{\widehat{m}} \xrightarrow{p} \sigma_{12}$, $\widehat{\sigma}_{22}^{\widehat{m}} \xrightarrow{p} \sigma_{22}$ as $(l_{sn}, \widehat{m}) \to \infty$. Hence, when the underlying Gaussian random field has absolutely summable covariance functions, $\widehat{\sigma}_{11}^{\widehat{m}} \approx \sigma_{11}$, $\widehat{\sigma}_{12}^{\widehat{m}} \approx \sigma_{12}$, $\widehat{\sigma}_{22}^{\widehat{m}} \approx \sigma_{22}$ when l_{sn} and $\widehat{m}$ are sufficiently large.

A Appendix

A.1 Proofs of Asymptotic Variance Σ^m

A.1.1 Proof of Theorem 2.2.1

Proof.

$$\begin{aligned}\text{Var}\left[\widehat{\gamma}_0(1/q)\right] &= \text{Var}\left\{\frac{1}{2(q-1)|A|}\sum_{i=1}^{|A|}\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2\right\} \\ &= \frac{1}{4(q-1)^2|A|^2}\sum_{i=1}^{|A|}\sum_{s=1}^{|A|}\text{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2\right].\end{aligned}$$

Since $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ is a second-order stationary Gaussian process, the joint distribution of $\{Z_{ij}-Z_{i(j+1)} : i \in A, j = 1, 2, ..., q-1\}$ is multivariate normal. The value of $\text{Cov}[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,$ $\sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2]$ depends on the relative positions of $(Z_{i1}, ..., Z_{iq})$ and $(Z_{s1}, ..., Z_{sq})$. This will be discussed in four scenarios.

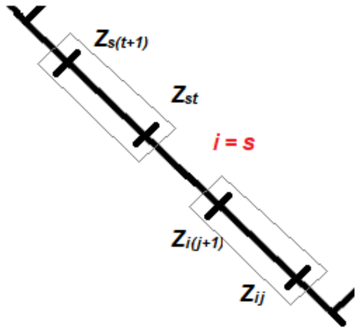
I. $i = s$

$$\begin{aligned}&\text{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2\right] \\ =&\text{Var}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2\right] \\ =&\sum_{j=1}^{q-1}\sum_{t=1}^{q-1}\text{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{it}-Z_{i(t+1)}\right)^2\right].\end{aligned}$$

By the properties of the moments of a multivariate normal random vector, when $j = t$,

$$\text{Var}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2\right] = 2\left[2\gamma_0(1/q)\right]^2.$$

When $j \neq t$, assume that $d(Z_{ij}, Z_{it}) = k/q$, $k \in \{1, 2, ..., (qm+1) \wedge (q-2)\}$. By the properties of the moments of a multivariate normal random vector,



$$\mathrm{Var}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2\right]=\mathrm{Var}\left[\left(Z_{it}-Z_{i(t+1)}\right)^2\right]=2\left[2\gamma_0(1/q)\right]^2,$$

and

$$\mathrm{Corr}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{it}-Z_{i(t+1)}\right)^2\right]=\left[\mathrm{Corr}\left(Z_{ij}-Z_{i(j+1)},Z_{it}-Z_{i(t+1)}\right)\right]^2.$$

- If $k=1$

$$\begin{aligned}&\mathrm{Corr}\left(Z_{ij}-Z_{i(j+1)},Z_{it}-Z_{i(t+1)}\right)\\=&\frac{\mathrm{Cov}\left(Z_{ij}-Z_{i(j+1)},Z_{it}-Z_{i(t+1)}\right)}{2\gamma_0(1/q)}\\=&\frac{\mathrm{Cov}\left(Z_{ij},Z_{st}\right)-\mathrm{Cov}\left(Z_{ij},Z_{i(t+1)}\right)-\mathrm{Cov}\left(Z_{i(j+1)},Z_{it}\right)+\mathrm{Cov}\left(Z_{i(j+1)},Z_{i(t+1)}\right)}{2\gamma_0(1/q)}\\=&\frac{C_{fc}\left(1/q\right)-C_{fc}\left(2/q\right)-\sigma^2+C_{fc}\left(1/q\right)}{2\gamma_0(1/q)}\\=&\frac{-2\gamma_0(1/q)+\gamma_c(2/q)}{2\gamma_0(1/q)},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{it}-Z_{i(t+1)}\right)^2\right]&=2\left[2\gamma_0(1/q)\right]^2\cdot\left[\frac{-2\gamma_0(1/q)+\gamma_c(2/q)}{2\gamma_0(1/q)}\right]^2\\&=2\left[2\gamma_0(1/q)-\gamma_c(2/q)\right]^2.\end{aligned}$$

- If $k=2$

$$\mathrm{Corr}\left(Z_{ij}-Z_{i(j+1)},Z_{it}-Z_{i(t+1)}\right)=\frac{\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)}{2\gamma_0(1/q)},$$

and

$$\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{it} - Z_{i(t+1)}\right)^2\right] = 2\left[2\gamma_0(1/q)\right]^2 \cdot \left[\frac{\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)}{2\gamma_0(1/q)}\right]^2$$
$$= 2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.$$

- If $k \geq 3$

$$\text{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{it} - Z_{i(t+1)}\right) = \frac{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]}{2\gamma_0(1/q)},$$

and

$$\begin{aligned}
&\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{it} - Z_{i(t+1)}\right)^2\right] \\
=&2\left[2\gamma_0(1/q)\right]^2 \cdot \left\{\frac{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]}{2\gamma_0(1/q)}\right\}^2 \\
=&2\Big\{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]\Big\}^2.
\end{aligned}$$

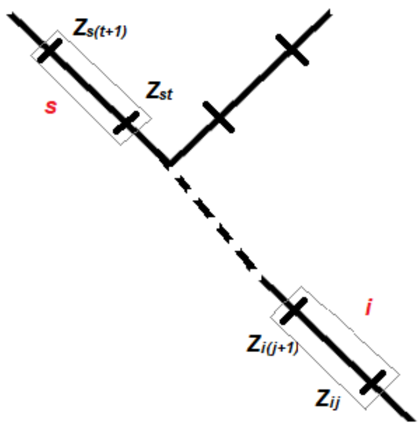
The cardinalities of $\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right)$ pairs by k within one segment can be easily shown as $(q-k-1) \vee 0$. Moreover, the number of segments is $|A|$. Thus, the summation of $\text{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij} - Z_{i(j+1)}\right)^2, \sum_{t=1}^{q-1}\left(Z_{st} - Z_{s(t+1)}\right)^2\right]$ under this scenario is

$$\begin{aligned}
&|A|(q-1) \cdot 2\left[2\gamma_0(1/q)\right]^2 + 2|A|(q-2) \cdot 2\left[2\gamma_0(1/q) - \gamma_c(2/q)\right]^2 \\
&+ 2|A|[(q-3) \vee 0] \cdot 2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ \sum_{k=3}^{(qm+1)\wedge(q-2)} 2|A|(q-k-1) \cdot 2\Big\{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]\Big\}^2 \\
=&2|A|(q-1)\left[2\gamma_0(1/q)\right]^2 + 4|A|(q-2)\left[2\gamma_0(1/q) - \gamma_c(2/q)\right]^2 \\
&+ 4|A|[(q-3) \vee 0]\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ 4|A| \sum_{k=3}^{(qm+1)\wedge(q-2)} (q-k-1)\Big\{\gamma_c[(k-1)/q] - 2\gamma_c(k/q) + \gamma_c[(k+1)/q]\Big\}^2.
\end{aligned}$$

II. Segments i, s are FC, $d(Z_{i1}, Z_{s1}) \leq \lceil m \rceil \leq m + 1/q$

$$\text{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij} - Z_{i(j+1)}\right)^2, \sum_{t=1}^{q-1}\left(Z_{st} - Z_{s(t+1)}\right)^2\right]$$
$$= \sum_{j=1}^{q-1}\sum_{t=1}^{q-1}\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right].$$

Assume that $d(Z_{i1}, Z_{s1}) = k$, $k \in \{1, ..., \lceil m \rceil\}$. For fixed segments i, s with $d(Z_{i1}, Z_{s1}) = k$, let $d(Z_{ij}, Z_{st}) = l/q$, $l \in \{q(k-1)+2, ..., q(k+1)-2\}$. By the properties of the moments of a multivariate normal random vector,



$$\text{Var}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2\right] = \text{Var}\left[\left(Z_{st} - Z_{s(t+1)}\right)^2\right] = 2\left[2\gamma_0(1/q)\right]^2,$$

and

$$\text{Corr}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right] = \left[\text{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right)\right]^2.$$

- If $l = 2$

Without loss of generality, assume that $(i,s) \in B$, $j = q-1$ and $t = 1$, then

$$\begin{aligned}
&\operatorname{Corr}\left(Z_{i(q-1)} - Z_{iq}, Z_{s1} - Z_{s2}\right) \\
=&\frac{\operatorname{Cov}\left(Z_{i(q-1)} - Z_{iq}, Z_{s1} - Z_{s2}\right)}{2\gamma_0(1/q)} \\
=&\frac{\operatorname{Cov}\left(Z_{i(q-1)}, Z_{s1}\right) - \operatorname{Cov}\left(Z_{i(q-1)}, Z_{s2}\right) - \operatorname{Cov}\left(Z_{iq}, Z_{s1}\right) + \operatorname{Cov}\left(Z_{iq}, Z_{s2}\right)}{2\gamma_0(1/q)} \\
=&\frac{-C_{fc}\left(1/q\right) + 2C_{fc}\left(2/q\right) - C_{fc}\left(3/q\right)}{2\gamma_0(1/q)} \\
=&\frac{\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)}{2\gamma_0(1/q)},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Cov}\left[\left(Z_{i(q-1)} - Z_{iq}\right)^2, \left(Z_{s1} - Z_{s2}\right)^2\right] &= 2\left[2\gamma_0(1/q)\right]^2 \cdot \left[\frac{\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)}{2\gamma_0(1/q)}\right]^2 \\
&= 2\left[\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.
\end{aligned}$$

- If $l > 2$

$$\begin{aligned}
&\operatorname{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right) \\
=&\frac{\operatorname{Cov}\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right)}{2\gamma_0(1/q)} \\
=&\frac{\operatorname{Cov}\left(Z_{ij}, Z_{st}\right) - \operatorname{Cov}\left(Z_{ij}, Z_{s(t+1)}\right) - \operatorname{Cov}\left(Z_{i(j+1)}, Z_{st}\right) + \operatorname{Cov}\left(Z_{i(j+1)}, Z_{s(t+1)}\right)}{2\gamma_0(1/q)} \\
=&\frac{-C_{fc}\left[(l-1)/q\right] + 2C_{fc}\left(l/q\right) - C_{fc}\left[(l+1)/q\right]}{2\gamma_0(1/q)} \\
=&\frac{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]}{2\gamma_0(1/q)},
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right] \\
=&2\left[2\gamma_0(1/q)\right]^2 \cdot \left\{\frac{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]}{2\gamma_0(1/q)}\right\}^2 \\
=&2\left\{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]\right\}^2.
\end{aligned}$$

The cardinalities of $(Z_{ij} - Z_{i(j+1)},\ Z_{st} - Z_{s(t+1)})$ pairs by l given fixed segments i, s with

$d(Z_{i1}, Z_{s1}) = k$ are shown in Table 9.

l	# of $\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right)$ Pairs
$q(k-1)+2$	1
$q(k-1)+3$	2
$\vdots$	$\vdots$
l	$l - q(k-1) - 1 = q - 1 - (qk - l)$
$\vdots$	$\vdots$
$qk-1$	$q-2$
qk	$q-1$
$qk+1$	$q-2$
$\vdots$	$\vdots$
l	$q(k+1) - l - 1 = q - 1 - (l - qk)$
$\vdots$	$\vdots$
$q(k+1)-2$	1

Table 9: # of $\left(Z_{ij} - Z_{i(j+1)}, Z_{st} - Z_{s(t+1)}\right)$ pairs by l given k

Thus, $\sum_{j=1}^{q-1} \sum_{t=1}^{q-1} \text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right]$ given fixed segments i, s with $d(Z_{i1}, Z_{s1}) = k$ is

$$\sum_{l=q(k-1)+2}^{q(k+1)-2} [q - 1 - (l - qk) \vee (qk - l)] \cdot 2\Big\{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]\Big\}^2,$$

or, when $k = 1$,

$$2\left[\gamma_1(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 + \sum_{l=3}^{2q-2} [q - 1 - (l - q) \vee (q - l)] \cdot 2\Big\{\gamma_c[(l-1)/q] - 2\gamma_c(l/q) + \gamma_c[(l+1)/q]\Big\}^2.$$

k	# of Segments (i, s) Pairs
1	$2(\lvert A\rvert - 1)$
2	$2(\lvert A\rvert - 3)$
$\vdots$	$\vdots$
k	$2(\lvert A\rvert - 2^k + 1)$
$\vdots$	$\vdots$
$\lceil m\rceil$	$2(\lvert A\rvert - 2^{\lceil m\rceil} + 1)$

Table 10: # of segments (i, s) pairs by k when i, s are FC

The cardinalities of segments (i, s) pairs by k under the scenario that i, s are FC and $d(Z_{i1}, Z_{s1}) \leq$

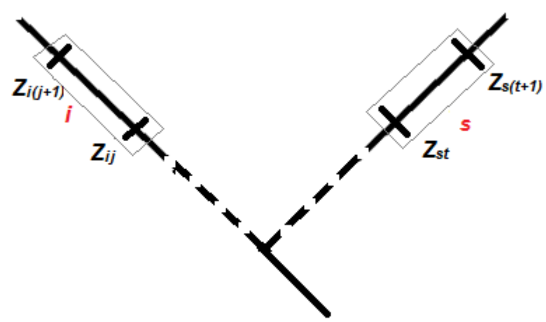
$\lceil m \rceil$ are shown in Table 10. Thus, the summation of $\text{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2, \sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2\right]$ under this scenario is

$$\begin{aligned}
&2(|A|-1)\cdot 2\left[\gamma_1(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2 \\
&+2(|A|-1)\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\cdot 2\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2 \\
&+\sum_{k=2}^{\lceil m\rceil}2(|A|-2^k+1)\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\cdot 2\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2 \\
=&4(|A|-1)\left[\gamma_1(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2 \\
&+4(|A|-1)\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2 \\
&+4\sum_{k=2}^{\lceil m\rceil}(|A|-2^k+1)\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2.
\end{aligned}$$

III. Segments i, s are FU, and for (Z_{i1}, Z_{s1}), $\max\{a,b\} \le \lfloor m \rfloor + 1/2q < m + 1/q$

Assume that for pair (Z_{i1}, Z_{s1}), $a = k + 1/2q$ and $b = l + 1/2q$, $k, l \in \{0, 1, 2, ..., \lfloor m \rfloor\}$. For fixed segments i, s such that for pair (Z_{i1}, Z_{s1}), $a = k + 1/2q$ and $b = l + 1/2q$, let the distances-to-common-junction for (Z_{ij}, Z_{st}) be $a = u/q + 1/2q, b = v/q + 1/2q$, $u \in \{qk, ..., q(k+1)-2\}$ and $v \in \{ql, ..., q(l+1)-2\}$. Similar to scenario II,

$$\text{Var}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2\right] = \text{Var}\left[\left(Z_{st}-Z_{s(t+1)}\right)^2\right] = 2\left[2\gamma_0(1/q)\right]^2.$$



We have

$$
\begin{aligned}
&\operatorname{Corr}\left(Z_{ij}-Z_{i(j+1)}, Z_{st}-Z_{s(t+1)}\right)\\
=&\frac{\operatorname{Cov}\left(Z_{ij}-Z_{i(j+1)}, Z_{st}-Z_{s(t+1)}\right)}{2\gamma_0(1/q)}\\
=&\frac{\operatorname{Cov}\left(Z_{ij}, Z_{st}\right)-\operatorname{Cov}\left(Z_{ij}, Z_{s(t+1)}\right)-\operatorname{Cov}\left(Z_{i(j+1)}, Z_{st}\right)+\operatorname{Cov}\left(Z_{i(j+1)}, Z_{s(t+1)}\right)}{2\gamma_0(1/q)}\\
=&\frac{1}{2\gamma_0(1/q)}\Big\{C_{fu}(a\wedge b, a\vee b)-C_{fu}[a\wedge(b+1/q), a\vee(b+1/q)]\\
&\qquad -C_{fu}[(a+1/q)\wedge b, (a+1/q)\vee b]\\
&\qquad +C_{fu}[(a+1/q)\wedge(b+1/q), (a+1/q)\vee(b+1/q)]\Big\}\\
=&\frac{1}{2\gamma_0(1/q)}\Big\{-\gamma_u(a\wedge b, a\vee b)+\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]\\
&\qquad +\gamma_u[(a+1/q)\wedge b, (a+1/q)\vee b]\\
&\qquad -\gamma_u[(a+1/q)\wedge(b+1/q), (a+1/q)\vee(b+1/q)]\Big\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{st}-Z_{s(t+1)}\right)^2\right]\\
=&2\left[2\gamma_0(1/q)\right]^2\left\{\frac{1}{2\gamma_0(1/q)}\Big[-\gamma_u(a\wedge b, a\vee b)+\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]\right.\\
&\qquad +\gamma_u[(a+1/q)\wedge b, (a+1/q)\vee b]\\
&\qquad \left.-\gamma_u[(a+1/q)\wedge(b+1/q), (a+1/q)\vee(b+1/q)]\Big]\right\}^2\\
=&2\Big\{\gamma_u(a\wedge b, a\vee b)-\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]-\gamma_u[(a+1/q)\wedge b, (a+1/q)\vee b]\\
&\quad +\gamma_u[(a+1/q)\wedge(b+1/q), (a+1/q)\vee(b+1/q)]\Big\}^2. \qquad \text{(a.1.1)}
\end{aligned}
$$

1) If $a=b$, i.e., $k=l$ and $u=v$

When $k = l$, $a = b$, then by (a.1.1),

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right] \\
=&2\Big[\gamma_u(a,a) - \gamma_u(a, a+1/q) - \gamma_u(a, a+1/q) + \gamma_u(a+1/q, a+1/q)\Big]^2 \\
=&2\Big\{\gamma_u[(2u+1)/2q, (2u+1)/2q] - 2\gamma_u[(2u+1)/2q, (2u+3)/2q] \\
&\quad + \gamma_u[(2u+3)/2q, (2u+3)/2q]\Big\}^2.
\end{aligned}
$$

2) If $a > b$, i.e., $a \geq b + 1/q$

When $a \geq b + 1/q$, $b < b + 1/q \leq a < a + 1/q$, then by (a.1.1),

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right] \\
=&2\Big[\gamma_u(b,a) - \gamma_u(b+1/q, a) - \gamma_u(b, a+1/q) + \gamma_u(b+1/q, a+1/q)\Big]^2 \\
=&2\Big\{\gamma_u[(2v+1)/2q, (2u+1)/2q] - \gamma_u[(2v+3)/2q, (2u+1)/2q] \\
&\quad - \gamma_u[(2v+1)/2q, (2u+3)/2q] + \gamma_u[(2v+3)/2q, (2u+3)/2q]\Big\}^2.
\end{aligned}
$$

3) If $a < b$, i.e., $a \leq b - 1/q$

When $a \leq b - 1/q$, $a < a + 1/q \leq b < b + 1/q$, then by (a.1.1),

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right] \\
=&2\Big[\gamma_u(a,b) - \gamma_u(a, b+1/q) - \gamma_u(a+1/q, b) + \gamma_u(a+1/q, b+1/q)\Big]^2 \\
=&2\Big\{\gamma_u[(2u+1)/2q, (2v+1)/2q] - \gamma_u[(2u+1)/2q, (2v+3)/2q] \\
&\quad - \gamma_u[(2u+3)/2q, (2v+1)/2q] + \gamma_u[(2u+3)/2q, (2v+3)/2q]\Big\}^2.
\end{aligned}
$$

Thus, $\sum_{j=1}^{q-1}\sum_{t=1}^{q-1} \operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{st} - Z_{s(t+1)}\right)^2\right]$ given fixed ixed segments i, s such that for pair (Z_{i1}, Z_{s1}), $a = k + 1/2q$ and $b = l + 1/2q$ is

- when $k = l$

$$\sum_{u=qk}^{q(k+1)-2} 2\Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q] - 2\gamma_u[(2u+1)/2q,(2u+3)/2q] + \gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2$$
$$+ 2\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1} 2\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q] - \gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$- \gamma_u[(2v+1)/2q,(2u+3)/2q] + \gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2,$$

- when $k > l$

$$\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2} 2\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q] - \gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$- \gamma_u[(2v+1)/2q,(2u+3)/2q] + \gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2,$$

- when $k < l$

$$\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2} 2\Big\{\gamma_u[(2u+1)/2q,(2v+1)/2q] - \gamma_u[(2u+3)/2q,(2v+1)/2q]$$
$$- \gamma_u[(2u+1)/2q,(2v+3)/2q] + \gamma_u[(2u+3)/2q,(2v+3)/2q]\Big\}^2.$$

The cardinalities of segments (i,s) pairs by (k,l) under the scenario that segments i,j are FU is tabulated in Table 11.

$k\backslash l$	0	1	2	$\cdots$	l	$\cdots$	$\lfloor m\rfloor$
0	$\lvert A\rvert-1$	$\lvert A\rvert-3$	$\lvert A\rvert-7$	$\cdots$	$\lvert A\rvert-2^{l+1}+1$	$\cdots$	$\lvert A\rvert-2^{\lfloor m\rfloor+1}+1$
1	$\lvert A\rvert-3$	$2(\lvert A\rvert-3)$	$2(\lvert A\rvert-7)$	$\cdots$	$2(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$
2	$\lvert A\rvert-7$	$2(\lvert A\rvert-7)$	$2^2(\lvert A\rvert-7)$	$\cdots$	$2^2(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2^2(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
k	$\lvert A\rvert-2^{k+1}+1$	$2(\lvert A\rvert-2^{k+1}+1)$	$2^2(\lvert A\rvert-2^{k+1}+1)$	$\cdots$	$2^{k\wedge l}(\lvert A\rvert-2^{k\vee l+1}+1)$	$\cdots$	$2^k(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\lfloor m\rfloor$	$\lvert A\rvert-2^{\lfloor m\rfloor+1}+1$	$2(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$	$2^2(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$	$\cdots$	$2^l(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$	$\cdots$	$2^{\lfloor m\rfloor}(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)$

Table 11: # of segment (i,s) pairs by (k,l) when i,s are FU

Thus, the summation of $\mathrm{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2, \sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2\right]$ under this scenario is

$$\sum_{k=0}^{\lfloor m\rfloor} 2^k(|A|-2^{k+1}+1)\cdot$$
$$\sum_{u=qk}^{q(k+1)-2} 2\Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q]-2\gamma_u[(2u+1)/2q,(2u+3)/2q]+\gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2$$
$$+\sum_{k=0}^{\lfloor m\rfloor} 2^k(|A|-2^{k+1}+1)\cdot$$
$$2\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1} 2\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2v+1)/2q,(2u+3)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2$$

$$+2\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=0}^{k-1} 2^l(|A|-2^{k+1}+1)\cdot$$
$$\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2} 2\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2v+1)/2q,(2u+3)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2$$
$$=\sum_{k=0}^{\lfloor m\rfloor} 2^{k+1}(|A|-2^{k+1}+1)\cdot$$
$$\sum_{u=qk}^{q(k+1)-2} \Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q]-2\gamma_u[(2u+1)/2q,(2u+3)/2q]+\gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2$$
$$+\sum_{k=0}^{\lfloor m\rfloor} 2^{k+2}(|A|-2^{k+1}+1)\cdot$$
$$\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1} \Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2v+1)/2q,(2u+3)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2$$
$$+\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=0}^{k-1} 2^{l+2}(|A|-2^{k+1}+1)\cdot$$
$$\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2} \Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2v+1)/2q,(2u+3)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2.$$

IV. Otherwise

In this scenario, the random vectors $(Z_{i1},...,Z_{iq})$ and $(Z_{s1},...,Z_{sq})$ are uncorrelated, thus

$$\mathrm{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\sum_{t=1}^{q-1}\left(Z_{st}-Z_{s(t+1)}\right)^2\right]=0.$$

Summarizing scenarios I-IV, we have

$$\begin{aligned}
&|A|\mathrm{Var}\left[\widehat{\gamma}_0(1/q)\right]\\
=&\frac{1}{4(q-1)^2|A|}\Bigg\{2|A|(q-1)\left[2\gamma_0(1/q)\right]^2+4|A|(q-2)\left[2\gamma_0(1/q)-\gamma_c(2/q)\right]^2\\
&\qquad+4|A|[(q-3)\vee 0]\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&\qquad+4|A|\sum_{k=3}^{(qm+1)\wedge(q-2)}(q-k-1)\Big\{\gamma_c[(k-1)/q]-2\gamma_c(k/q)+\gamma_c[(k+1)/q]\Big\}^2\\
&\qquad+4(|A|-1)\left[\gamma_1(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&\qquad+4(|A|-1)\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2\\
&\qquad+4\sum_{k=2}^{\lceil m\rceil}(|A|-2^k+1)\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2\\
&\qquad+\sum_{k=0}^{\lfloor m\rfloor}2^{k+1}(|A|-2^{k+1}+1)\cdot\\
&\qquad\quad\sum_{u=qk}^{q(k+1)-2}\Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q]-2\gamma_u[(2u+1)/2q,(2u+3)/2q]\\
&\qquad\qquad+\gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2\\
&\qquad+\sum_{k=0}^{\lfloor m\rfloor}2^{k+2}(|A|-2^{k+1}+1)\cdot\\
&\qquad\quad\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]\\
&\qquad\qquad-\gamma_u[(2v+1)/2q,(2u+3)/2q]-\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2\\
&\qquad+\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=0}^{k-1}2^{l+2}(|A|-2^{k+1}+1)\cdot\\
&\qquad\quad\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+3)/2q,(2u+1)/2q]\\
&\qquad\qquad-\gamma_u[(2v+1)/2q,(2u+3)/2q]-\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2(q-1)}\left[2\gamma_0(1/q)\right]^2+\frac{q-2}{(q-1)^2}\left[2\gamma_0(1/q)-\gamma_c(2/q)\right]^2+\frac{(q-3)\vee 0}{(q-1)^2}\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+\frac{1}{(q-1)^2}\sum_{k=3}^{(qm+1)\wedge(q-2)}(q-k-1)\Big\{\gamma_c[(k-1)/q]-2\gamma_c(k/q)+\gamma_c[(k+1)/q]\Big\}^2\\
&+\frac{|A|-1}{(q-1)^2|A|}\left[\gamma_1(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+\frac{|A|-1}{(q-1)^2|A|}\sum_{l=3}^{2q-2}[q-1-(l-q)\vee(q-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2\\
&+\frac{1}{(q-1)^2}\sum_{k=2}^{\lceil m\rceil}\frac{|A|-2^k+1}{|A|}\sum_{l=q(k-1)+2}^{q(k+1)-2}[q-1-(l-qk)\vee(qk-l)]\Big\{\gamma_c[(l-1)/q]-2\gamma_c(l/q)+\gamma_c[(l+1)/q]\Big\}^2\\
&+\frac{1}{(q-1)^2}\sum_{k=0}^{\lfloor m\rfloor}\frac{2^{k-1}(|A|-2^{k+1}+1)}{|A|}\sum_{u=qk}^{q(k+1)-2}\Big\{\gamma_u[(2u+1)/2q,(2u+1)/2q]-2\gamma_u[(2u+1)/2q,(2u+3)/2q]\\
&\qquad+\gamma_u[(2u+3)/2q,(2u+3)/2q]\Big\}^2\\
&+\frac{1}{(q-1)^2}\sum_{k=0}^{\lfloor m\rfloor}\frac{2^k(|A|-2^{k+1}+1)}{|A|}\cdot\\
&\qquad\sum_{u=qk+1}^{q(k+1)-2}\sum_{v=qk}^{u-1}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+1)/2q,(2u+3)/2q]\\
&\qquad-\gamma_u[(2v+3)/2q,(2u+1)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2\\
&+\frac{1}{(q-1)^2}\sum_{k=1}^{\lfloor m\rfloor}\sum_{l=0}^{k-1}\frac{2^l(|A|-2^{k+1}+1)}{|A|^2}\cdot\\
&\qquad\sum_{u=qk}^{q(k+1)-2}\sum_{v=ql}^{q(l+1)-2}\Big\{\gamma_u[(2v+1)/2q,(2u+1)/2q]-\gamma_u[(2v+1)/2q,(2u+3)/2q]\\
&\qquad-\gamma_u[(2v+3)/2q,(2u+1)/2q]+\gamma_u[(2v+3)/2q,(2u+3)/2q]\Big\}^2.
\end{aligned}
$$

The facts that $l_{sn}\to\infty$ implies $|A|\to\infty$ and $\lim_{|A|\to\infty}\frac{|A|-C}{|A|}=1$ for any $C<\infty$ complete the proof of Theorem 2.2.1. □

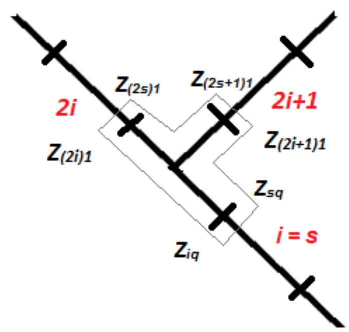
A.1.2 Proof of Theorem 2.2.2

Proof.

$$
\begin{aligned}
\mathrm{Var}\left[\widehat{\gamma}_1(1/q)\right]&=\mathrm{Var}\left\{\frac{1}{2(|A|-1)}\sum_{i=1}^{(|A|-1)/2}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\right\}\\
&=\frac{1}{4(|A|-1)^2}\sum_{i=1}^{(|A|-1)/2}\sum_{s=1}^{(|A|-1)/2}\mathrm{Cov}\Big[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\\
&\qquad\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\Big].
\end{aligned}
$$

Similarly, since $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ is a second-order Gaussian process, the joint distribution of $\{Z_{iq} - Z_{k1} : (i, k) \in B\}$ is multivariate normal. The value of each summand depends on the relative positions of the six sites involved. This will be discussed in four scenarios.

I. $i = s$



By the properties of the moments of a multivariate normal random vector,

$$\mathrm{Var}\left[\left(Z_{iq} - Z_{(2i)1}\right)^2\right] = \mathrm{Var}\left[\left(Z_{iq} - Z_{(2i+1)1}\right)^2\right] = 2\left[2\gamma_1(1/q)\right]^2,$$

and

$$\mathrm{Corr}\left[\left(Z_{iq} - Z_{(2i)1}\right)^2, \left(Z_{iq} - Z_{(2i+1)1}\right)^2\right] = \left[\mathrm{Corr}\left(Z_{iq} - Z_{(2i)1}, Z_{iq} - Z_{(2i+1)1}\right)\right]^2.$$

We have

$$\begin{aligned}
&\mathrm{Corr}\left(Z_{iq} - Z_{(2i)1}, Z_{iq} - Z_{(2i+1)1}\right) \\
=&\frac{\mathrm{Cov}\left(Z_{iq} - Z_{(2i)1}, Z_{iq} - Z_{(2i+1)1}\right)}{2\gamma_1(1/q)} \\
=&\frac{\sigma^2 - \mathrm{Cov}\left(Z_{iq}, Z_{(2i+1)1}\right) - \mathrm{Cov}\left(Z_{(2i)1}, Z_{iq}\right) + \mathrm{Cov}\left(Z_{(2i)1}, Z_{(2i+1)1}\right)}{2\gamma_1(1/q)} \\
=&\frac{2\gamma_1(1/q) - \gamma_u(1/2q, 1/2q)}{2\gamma_1(1/q)}.
\end{aligned}$$

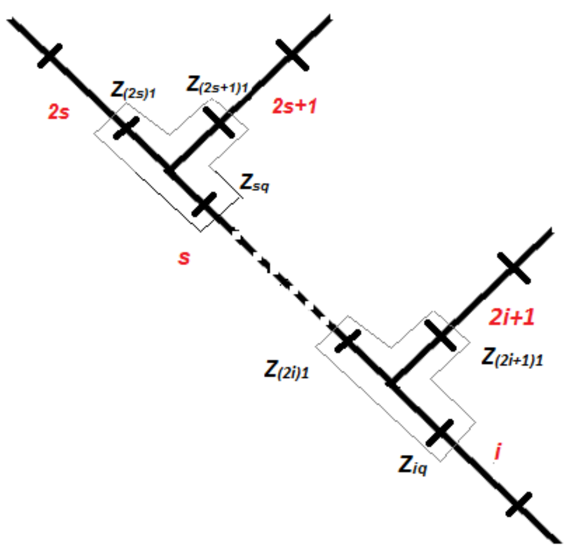
Thus

$$
\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&\mathrm{Var}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&\mathrm{Var}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2\right]+\mathrm{Var}\left[\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
&+2\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&4\left[2\gamma_1(1/q)\right]^2+4\left[2\gamma_1(1/q)\right]^2\cdot\left[\frac{2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)}{2\gamma_1(1/q)}\right]^2\\
=&4\left[2\gamma_1(1/q)\right]^2+4\left[2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)\right]^2.
\end{aligned}
$$

It can also be shown that the cardinalities of this scenario is $(|A|-1)/2$.

II. Segments i, s are FC, $d(Z_{iq}, Z_{sq}) \leq m + 1/p$

$$
\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2\right]+\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
&+\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2\right]+\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right].
\end{aligned}
$$



Assume $d(Z_{iq}, Z_{sq}) = k$, $k \in \{1, 2, ..., \lfloor m+1/q \rfloor\}$. Without loss of generality, assume that segment $2i$ and s are FC. By the properties of the moments of a multivariate normal random vector,

1) If $k \geq 2$

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[2\gamma_1(1/q)\right]^2\cdot\left[\frac{-\gamma_c(k+1/q)+2\gamma_c(k)-\gamma_c(k-1/q)}{2\gamma_1(1/q)}\right]^2\\
=&2\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2,\\
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[2\gamma_1(1/q)\right]^2\cdot\left[\frac{-\gamma_c(k)+\gamma_c(k+1/q)+\gamma_u(1/2q,k-1/2q)-\gamma_u(1/2q,k+1/2q)}{2\gamma_1(1/q)}\right]^2\\
=&2\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&4\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2\\
&+4\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2.
\end{aligned}$$

2) If $k = 1$ and $q = 2$

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i)1}\right)^2,\left(Z_{s2}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i)1}\right)^2,\left(Z_{s2}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_0(0.5)-2\gamma_c(1)+\gamma_c(1.5)\right]^2,\\
&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i+1)1}\right)^2,\left(Z_{s2}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i+1)1}\right)^2,\left(Z_{s2}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_c(1)-\gamma_c(1.5)-\gamma_u(0.25,0.75)+\gamma_u(0.25,1.25)\right]^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i)1}\right)^2+\left(Z_{i2}-Z_{(2i+1)1}\right)^2,\left(Z_{s2}-Z_{(2s)1}\right)^2+\left(Z_{s2}-Z_{(2s+1)1}\right)^2\right]\\
=&4\left[\gamma_0(0.5)-2\gamma_c(1)+\gamma_c(1.5)\right]^2+4\left[\gamma_c(1)-\gamma_c(1.5)-\gamma_u(0.25,0.75)+\gamma_u(0.25,1.25)\right]^2.
\end{aligned}$$

3) If $k=1$ and $q>2$

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i)1}\right)^2,\left(Z_{s2}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i)1}\right)^2,\left(Z_{s2}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_c(1-1/q)-2\gamma_c(1)+\gamma_c(1+1/q)\right]^2,\\
&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i+1)1}\right)^2,\left(Z_{s2}-Z_{(2s)1}\right)^2\right]\\
=&\mathrm{Cov}\left[\left(Z_{i2}-Z_{(2i+1)1}\right)^2,\left(Z_{s2}-Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_c(1)-\gamma_c(1+1/q)-\gamma_u(1/2q,1-1/2q)+\gamma_u(1/2q,1+1/2q)\right]^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&4\left[\gamma_c(1-1/q)-2\gamma_c(1)+\gamma_c(1+1/q)\right]^2\\
&+4\left[\gamma_c(1)-\gamma_c(1+1/q)-\gamma_u(1/2q,1-1/2q)+\gamma_u(1/2q,1+1/2q)\right]^2.
\end{aligned}$$

The cardinalities of $\left(Z_{iq},Z_{(2i)1},Z_{(2i+1)1}\right)$ $\left(Z_{sq},Z_{(2s)1},Z_{(2s+1)1}\right)$ pairs by k under the scenario that segments i,s are FC is tabulated in Table 12.

k	# of $\left(Z_{iq},Z_{(2i)1},Z_{(2i+1)1}\right)$ $\left(Z_{sq},Z_{(2s)1},Z_{(2s+1)1}\right)$ Pairs
1	$2[(\|A\|-1)/2-1]=\|A\|-3$
2	$2[(\|A\|-1)/2-3]=\|A\|-7$
$\vdots$	$\vdots$
k	$2\left[(\|A\|-1)/2-2^k+1\right]=\|A\|-2^{k+1}+1$
$\vdots$	$\vdots$
$\lfloor m+1/q\rfloor$	$\|A\|-2^{\lfloor m+1/q\rfloor+1}+1$

Table 12: # of $\left(Z_{i2},Z_{(2i)1},Z_{(2i+1)1}\right)$ $\left(Z_{sq},Z_{(2s)1},Z_{(2s+1)1}\right)$ pairs by k when i,s are FC

The summation of $\mathrm{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]$

under this scenario is

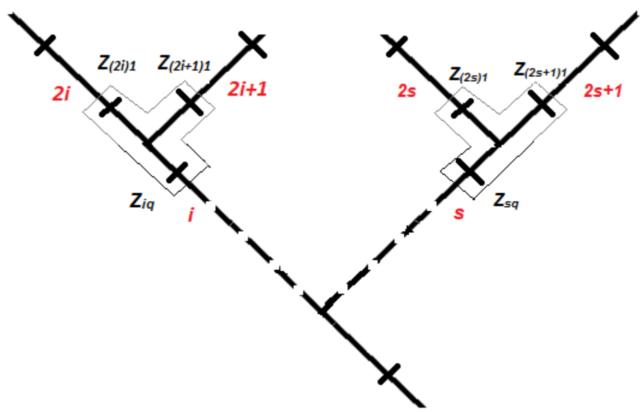
- when $q = 2$

$$
\begin{aligned}
&\mathrm{I}\Big\{\lfloor m+1/q \rfloor \geq 1\Big\} \cdot 4(|A|-3)\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+4\sum_{k=2}^{\lfloor m+1/q \rfloor} (|A|-2^{k+1}+1)\left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2 \\
&+4\sum_{k=1}^{\lfloor m+1/q \rfloor} (|A|-2^{k+1}+1)\left[\gamma_c(k) - \gamma_c(k+1/q) - \gamma_u(1/2q, k-1/2q) + \gamma_u(1/2q, k+1/2q)\right]^2,
\end{aligned}
$$

- when $q > 2$

$$
\begin{aligned}
&4\sum_{k=1}^{\lfloor m+1/q \rfloor} (|A|-2^{k+1}+1)\left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2 \\
&+4\sum_{k=1}^{\lfloor m+1/q \rfloor} (|A|-2^{k+1}+1)\left[\gamma_c(k) - \gamma_c(k+1/q) - \gamma_u(1/2q, k-1/2q) + \gamma_u(1/2q, k+1/2q)\right]^2.
\end{aligned}
$$

III. Segments i, s are FU, and for (Z_{iq}, Z_{sq}), $\max\{a, b\} \leq \lfloor m+1/q \rfloor - 1/2q < m + 1/q$



Assume for pair (Z_{iq}, Z_{jq}), $a = k - 1/2q$ and $b = l - 1/2q$, $k, l \in \{1, 2, ..., \lfloor m+1/q \rfloor\}$. Since

$$
\begin{aligned}
&\mathrm{Cov}\left(Z_{iq} - Z_{(2i)1}, Z_{jq} - Z_{(2j)1}\right) \\
=&\mathrm{Cov}\left(Z_{iq} - Z_{(2i)1}, Z_{jq} - Z_{(2j+1)1}\right) \\
=&\mathrm{Cov}\left(Z_{iq} - Z_{(2i+1)1}, Z_{jq} - Z_{(2j)1}\right) \\
=&\mathrm{Cov}\left(Z_{iq} - Z_{(2i+1)1}, Z_{jq} - Z_{(2j+1)1}\right),
\end{aligned}
$$

calculating the first term is sufficient to compute

$$\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(j+1)1}\right)^2,\left(Z_{jq}-Z_{(2j)1}\right)^2+\left(Z_{jq}-Z_{(2j+1)1}\right)^2\right].$$

We have

$$\begin{aligned}
&\operatorname{Cov}\left(Z_{iq}-Z_{(2i)1},Z_{jq}-Z_{(2j)1}\right)\\
=&-\gamma_u(a\wedge b,a\vee b)+\gamma_u[a\wedge(b+1/q),a\vee(b+1/q)]\\
&+\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]-\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)].
\end{aligned}$$

Thus

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2,\left(Z_{jq}-Z_{(2j)1}\right)^2\right]\\
=&2\left[2\gamma_1(1/q)\right]^2\cdot\\
&\left\{\frac{1}{2\gamma_1(1/q)}\Big[-\gamma_u(a\wedge b,a\vee b)+\gamma_u[a\wedge(b+1/q),a\vee(b+1/q)]\right.\\
&\qquad+\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]\\
&\qquad\left.-\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)]\Big]\right\}^2\\
=&2\Big\{\gamma_u(a\wedge b,a\vee b)-\gamma_u[a\wedge(b+1/q),a\vee(b+1/q)]-\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]\\
&+\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)]\Big\}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&8\Big\{\gamma_u(a\wedge b,a\vee b)-\gamma_u[a\wedge(b+1/q),a\vee(b+1/q)]-\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]\\
&+\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)]\Big\}^2. \qquad \text{(a.1.2)}
\end{aligned}$$

1) If $k=l$

When $k=l$, $a=b$, then by (a.1.2),

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&8\left[\gamma_u(a,a)-2\gamma_u(a,a+1/q)+\gamma_u(a+1/q,a+1/q)\right]^2\\
=&8\left[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)+\gamma_u(k+1/2q,k+1/2q)\right]^2.
\end{aligned}$$

2) If $k<l$, i.e., $k\le l-1$

When $k\le l-1$, $a<a+1/q<b<b+1/q$, then by (a.1.2),

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&8\left[\gamma_u(a,b)-\gamma_u(a,b+1/q)-\gamma_u(a+1/q,b)+\gamma_u(a+1/q,b+1/q)\right]^2\\
=&8\big[\gamma_u(k-1/2q,l-1/2q)-\gamma_u(k-1/2q,l+1/2q)-\gamma_u(k+1/2q,l-1/2q)\\
&\quad+\gamma_u(k+1/2q,l+1/2q)\big]^2.
\end{aligned}$$

3) If $k>l$, i.e., $k\ge l+1$

When $k\ge l+1$, $b<b+1/q<a<a+1/q$, then by (a.1.2),

$$\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2,\left(Z_{iq}-Z_{(2i)1}\right)^2+\left(Z_{iq}-Z_{(2i+1)1}\right)^2\right]\\
=&8\left[\gamma_u(b,a)-\gamma_u(b+1/q,a)-\gamma_u(b,a+1/q)+\gamma_u(b+1/q,a+1/q)\right]^2\\
=&8\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)-\gamma_u(l-1/2q,k+1/2q)\\
&\quad+\gamma_u(l+1/2q,k+1/2q)\big]^2.
\end{aligned}$$

$k\backslash l$	1	2	$\cdots$	l	$\cdots$	$\lfloor m+1/q\rfloor$
1	$(\lvert A\rvert-3)/2$	$(\lvert A\rvert-7)/2$	$\cdots$	$(\lvert A\rvert-2^{l+1}+1)/2$	$\cdots$	$(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)/2$
2	$(\lvert A\rvert-7)/2$	$\lvert A\rvert-7$	$\cdots$	$\lvert A\rvert-2^{l+1}+1$	$\cdots$	$\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1$
3	$(\lvert A\rvert-15)/2$	$\lvert A\rvert-15$	$\cdots$	$2(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
k	$(\lvert A\rvert-2^{k+1}+1)/2$	$\lvert A\rvert-2^{k+1}+1$	$\cdots$	$2^{k\wedge l-2}(\lvert A\rvert-2^{k\vee l+1}+1)$	$\cdots$	$2^{k-2}(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\lfloor m+1/q\rfloor$	$(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)/2$	$\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1$	$\cdots$	$2^{l-2}(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)$	$\cdots$	$2^{\lfloor m+1/q\rfloor-2}(\lvert A\rvert-2^{\lfloor m+1/q\rfloor+1}+1)$

Table 13: # of $\left(Z_{iq},Z_{(2i)1},Z_{(2i+1)1}\right)$, $\left(Z_{jq},Z_{(2j)1},Z_{(2j+1)1}\right)$ pairs by (k,l) when i,s are FU

The cardinalities of $\big(Z_{iq}, Z_{(2i)1}, Z_{(2i+1)1}\big)$ $\big(Z_{jq}, Z_{(2j)1}, Z_{(2j+1)1}\big)$ pairs by (k,l) under the scenario that segments i,s are FU is tabulated in Table 13. The summation of $\mathrm{Cov}\big[\,\big(Z_{iq}-Z_{(2i)1}\big)^2+\big(Z_{iq}-Z_{(2i+1)1}\big)^2, \big(Z_{sq}-Z_{(2s)1}\big)^2+\big(Z_{sq}-Z_{(2s+1)1}\big)^2\,\big]$ under this scenario is

$$
\begin{aligned}
&\sum_{k=1}^{\lfloor m+1/q\rfloor} 2^{k-2}(|A|-2^{k+1}+1)\cdot 8\left[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)+\gamma_u(k+1/2q,k+1/2q)\right]^2\\
&+2\sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1}2^{l-2}(|A|-2^{k+1}+1)\cdot 8\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)\\
&\qquad\qquad -\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\big]^2\\
=&\sum_{k=1}^{\lfloor m+1/q\rfloor} 2^{k+1}(|A|-2^{k+1}+1)\left[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)+\gamma_u(k+1/2q,k+1/2q)\right]^2\\
&+\sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1}2^{l+2}(|A|-2^{k+1}+1)\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)\\
&\qquad\qquad -\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\big]^2.
\end{aligned}
$$

IV. Otherwise

In this scenario, the random vectors $(Z_{iq}, Z_{(2i)1}, Z_{(2i+1)1})$ and $(Z_{iq}, Z_{(2j)1}, Z_{(2j+1)1})$ are uncorrelated, thus

$$\mathrm{Cov}\left[\big(Z_{iq}-Z_{(2i)1}\big)^2+\big(Z_{iq}-Z_{(2i+1)1}\big)^2, \big(Z_{iq}-Z_{(2i)1}\big)^2+\big(Z_{iq}-Z_{(2i+1)1}\big)^2\right]=0.$$

Summarizing I-IV, we have

- when $q=2$

$$
\begin{aligned}
&|A|\mathrm{Var}\left[\widehat{\gamma}_1(1/q)\right]\\
=&\frac{|A|}{4(|A|-1)^2}\sum_{i=1}^{(|A|-1)/2}\sum_{s=1}^{(|A|-1)/2}\mathrm{Cov}\Big[\,\big(Z_{iq}-Z_{(2i)1}\big)^2+\big(Z_{iq}-Z_{(2i+1)1}\big)^2,\\
&\qquad\qquad \big(Z_{sq}-Z_{(2s)1}\big)^2+\big(Z_{sq}-Z_{(2s+1)1}\big)^2\,\Big]\\
=&\frac{|A|}{4(|A|-1)^2}\Bigg\{\frac{|A|-1}{2}\cdot\left[4\left[2\gamma_1(1/q)\right]^2+4\left[2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)\right]^2\right]\\
&+\mathrm{I}\Big\{\lfloor m+1/q\rfloor\ge 1\Big\}\cdot 4(|A|-3)\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+4\sum_{k=2}^{\lfloor m+1/q\rfloor}(|A|-2^{k+1}+1)\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2\\
&+4\sum_{k=1}^{\lfloor m+1/q\rfloor}(|A|-2^{k+1}+1)\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{\lfloor m+1/q\rfloor} 2^{k+1}(|A|-2^{k+1}+1)[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)\\
&\qquad\qquad\qquad +\gamma_u(k+1/2q,k+1/2q)]^2\\
&+\sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1} 2^{l+2}(|A|-2^{k+1}+1)\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)\\
&\qquad\qquad\qquad -\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\big]^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{|A|}{2(|A|-1)}\left[2\gamma_1(1/q)\right]^2+\frac{|A|}{2(|A|-1)}\left[2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)\right]^2\\
&+\mathrm{I}\Big\{\lfloor m+1/q\rfloor\geq 1\Big\}\cdot\frac{|A|(|A|-3)}{(|A|-1)^2}\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+\sum_{k=2}^{\lfloor m+1/q\rfloor}\frac{|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2\\
&+\sum_{k=1}^{\lfloor m+1/q\rfloor}\frac{|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2\\
&+\sum_{k=1}^{\lfloor m+1/q\rfloor}\frac{2^{k-1}|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)\\
&\qquad\qquad\qquad +\gamma_u(k+1/2q,k+1/2q)]^2\\
&+\sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1}\frac{2^{l}|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)\\
&\qquad\qquad\qquad -\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\big]^2,
\end{aligned}
$$

- when $q>2$

$$
\begin{aligned}
&|A|\mathrm{Var}\left[\widehat{\gamma}_1(1/q)\right]\\
=&\frac{|A|}{2(|A|-1)}\left[2\gamma_1(1/q)\right]^2+\frac{|A|}{2(|A|-1)}\left[2\gamma_1(1/q)-\gamma_u(1/2q,1/2q)\right]^2\\
&+\sum_{k=1}^{\lfloor m+1/q\rfloor}\frac{|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2\\
&+\sum_{k=1}^{\lfloor m+1/q\rfloor}\frac{|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\left[\gamma_c(k)-\gamma_c(k+1/q)-\gamma_u(1/2q,k-1/2q)+\gamma_u(1/2q,k+1/2q)\right]^2\\
&+\sum_{k=1}^{\lfloor m+1/q\rfloor}\frac{2^{k-1}|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}[\gamma_u(k-1/2q,k-1/2q)-2\gamma_u(k-1/2q,k+1/2q)\\
&\qquad\qquad\qquad +\gamma_u(k+1/2q,k+1/2q)]^2\\
&+\sum_{k=2}^{\lfloor m+1/q\rfloor}\sum_{l=1}^{k-1}\frac{2^{l}|A|(|A|-2^{k+1}+1)}{(|A|-1)^2}\big[\gamma_u(l-1/2q,k-1/2q)-\gamma_u(l+1/2q,k-1/2q)\\
&\qquad\qquad\qquad -\gamma_u(l-1/2q,k+1/2q)+\gamma_u(l+1/2q,k+1/2q)\big]^2.
\end{aligned}
$$

The facts that $l_{sn}\to\infty$ implies $|A|\to\infty$ and $\lim_{|A|\to\infty}\frac{|A|-a}{|A|-b}=1$ for any $a,b\in\mathbb{R}$ complete the proof of Theorem 2.2.2. □

A.1.3 Proof of Theorem 2.2.3

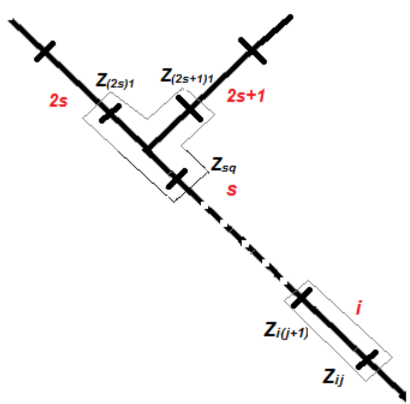
Proof.

$$
\begin{aligned}
&\mathrm{Cov}\left[\widehat{\gamma}_0(1/q), \widehat{\gamma}_1(1/q)\right] \\
=&\mathrm{Cov}\left\{ \frac{1}{2(q-1)|A|} \sum_{i=1}^{|A|} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2, \frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \left[\left(Z_{iq} - Z_{(2i)1}\right)^2 + \left(Z_{iq} - Z_{(2i+1)1}\right)^2 \right] \right\} \\
=&\frac{1}{4(q-1)|A|(|A|-1)} \sum_{i=1}^{|A|} \sum_{s=1}^{(|A|-1)/2} \mathrm{Cov}\left[\sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2 \right].
\end{aligned}
$$

Similarly, since $\{Z_{ij} : i \in A, j = 1, 2, ..., q\}$ is a second-order Gaussian process, the joint distribution of $\{Z_{is} - Z_{jt} : i, j \in A,\ s, t \in \{1, 2, ..., q\}\}$ is multivariate normal. The value of each summand depends on the relative positions of $(Z_{i1}, ..., Z_{iq})$ and $(Z_{sq}, Z_{(2s)q}, Z_{(2s+1)q})$. This will be discussed in four scenarios.

I. Segments i, s are FC, $i \leq s$, and $d(Z_{iq}, Z_{sq}) \leq \lfloor m \rfloor < m + 1/q$

$$
\begin{aligned}
&\mathrm{Cov}\left[\sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2 \right] \\
=&\sum_{j=1}^{q-1} \mathrm{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2 \right].
\end{aligned}
$$



Assume $d(Z_{iq}, Z_{sq}) = k$, $k \in \{0, 1, ..., \lfloor m \rfloor\}$. For fixed segments i, s with $d(Z_{iq}, Z_{sq}) = k$, let $d(Z_{ij}, Z_{sq}) = l/q$, $l \in \{qk+1, ..., q(k+1)-1\}$. By the properties of the moments of a multivariate normal random vector,

$$
\begin{aligned}
\operatorname{Var}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2\right] &= 2\left[2\gamma_0(1/q)\right]^2, \\
\operatorname{Var}\left[\left(Z_{sq}-Z_{(2s)1}\right)^2\right] &= \operatorname{Var}\left[\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] = 2\left[2\gamma_1(1/q)\right]^2.
\end{aligned}
$$

1) If $k = 0$ and $l = 1$

$$
\begin{aligned}
&\operatorname{Corr}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s)1}\right) \\
=&\frac{\operatorname{Cov}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s)1}\right)}{\sqrt{2\gamma_0(1/q)\cdot 2\gamma_1(1/q)}} \\
=&\frac{\operatorname{Cov}\left(Z_{ij}, Z_{sq}\right)-\operatorname{Cov}\left(Z_{ij}, Z_{(2s)1}\right)-\operatorname{Cov}\left(Z_{i(j+1)}, Z_{sq}\right)+\operatorname{Cov}\left(Z_{i(j+1)}, Z_{(2s)1}\right)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}} \\
=&\frac{-\gamma_0(1/q)+\gamma_c(2/q)-\gamma_1(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2\right] \\
=&\sqrt{2\left[2\gamma_0(1/q)\right]^2\cdot 2\left[2\gamma_1(1/q)\right]^2}\cdot\left[\frac{-\gamma_0(1/q)+\gamma_c(2/q)-\gamma_1(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\right]^2 \\
=&2\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2.
\end{aligned}
$$

Similarly,

$$
\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] = 2\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2.
$$

Hence

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&4\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2.
\end{aligned}
$$

2) If $k = 0$ and $l = 2$

$$
\begin{aligned}
&\operatorname{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s)1}\right) \\
=&\frac{C_{fc}\,(2/q) - C_{fc}\,(3/q) - C_{fc}\,(1/q) + C_{fc}\,(2/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}} \\
=&\frac{-2\gamma_c(2/q) + \gamma_c(3/q) + \gamma_0(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2\right] \\
=&\sqrt{2\left[2\gamma_0(1/q)\right]^2 \cdot 2\left[2\gamma_1(1/q)\right]^2} \cdot \left[\frac{-2\gamma_c(2/q) + \gamma_c(3/q) + \gamma_0(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\right]^2 \\
=&2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.
\end{aligned}
$$

Similarly,

$$
\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right] = 2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.
$$

Hence

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right] \\
=&4\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.
\end{aligned}
$$

3) If $l \geq 3$

$$
\begin{aligned}
&\operatorname{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s)1}\right) \\
=&\frac{C_{fc}\,(l/q) - C_{fc}\left[(l+1)/q\right] - C_{fc}\left[(l-1)/q\right] + C_{fc}\,(l/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}} \\
=&\frac{-2\gamma_c(l/q) + \gamma_c\left[(l+1)/q\right] + \gamma_c\left[(l-1)/q\right]}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2\right] \\
=&\sqrt{2\left\{2\gamma_0(1/q)\right]^2\cdot 2\left[2\gamma_1(1/q)\right]^2}\cdot\left\{\frac{-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]+\gamma_c\left[(l-1)/q\right]}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\right\}^2 \\
=&2\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2.
\end{aligned}
$$

Similarly,

$$
\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]=2\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2.
$$

Hence

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&4\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2.
\end{aligned}
$$

The cardinalities of (i,s) pairs by k under the scenario that segments i,s are FC and $i\le s$ is tabulated in Table 14.

k	# of Segments (i,s) Pairs
0	$(\lvert A\rvert-1)/2$
1	$(\lvert A\rvert-1)/2-1=(\lvert A\rvert-3)/2$
$\vdots$	$\vdots$
k	$(\lvert A\rvert-1)/2-(2^k-1)=(\lvert A\rvert-2^{k+1}+1)/2$
$\vdots$	$\vdots$
$\lfloor m\rfloor$	$(\lvert A\rvert-1)/2-(2^{\lfloor m\rfloor}-1)=(\lvert A\rvert-2^{\lfloor m\rfloor+1}+1)/2$

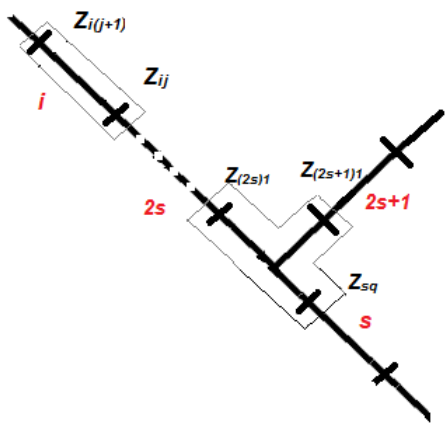
Table 14: # of segment (i,s) pairs by k when i,s are FC and $i\le s$

Thus, the summation of $\operatorname{Cov}\Big[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\Big]$ under this scenario is

$$
\begin{aligned}
&\frac{|A|-1}{2}\cdot 4\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2+\mathrm{I}\{q\geq 3\}\cdot\frac{|A|-1}{2}\cdot 4\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+\frac{|A|-1}{2}\sum_{l=3}^{q-1}4\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2\\
&+\sum_{k=1}^{\lfloor m\rfloor}\frac{|A|-2^{k+1}+1}{2}\sum_{l=qk+1}^{q(k+1)-1}4\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2\\
=&2(|A|-1)\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2+\mathrm{I}\{q\geq 3\}\cdot 2(|A|-1)\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+2(|A|-1)\sum_{l=3}^{q-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2\\
&+\sum_{k=1}^{\lfloor m\rfloor}2(|A|-2^{k+1}+1)\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2.
\end{aligned}
$$

II. Segments i, s are FC, $i > s$, and $d(Z_{sq}, Z_{i1}) \leq \lfloor m\rfloor + 1/q$

Without loss of generality, assume that segments i and $2s$ are also FC. Assume $d(Z_{(2s)1}, Z_{i1}) = k$, $k \in \{0, 1, ..., \lfloor m\rfloor\}$. For fixed segments i, s with $d(Z_{(2s)1}, Z_{i1}) = k$, let $d(Z_{(2s)1}, Z_{ij}) = l/q$, $l \in \{qk, ..., q(k+1) - 2\}$.



1) If $k = 0$ and $l = 0$

$$
\begin{aligned}
&\mathrm{Corr}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s)1}\right)\\
=&\frac{\mathrm{Cov}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s)1}\right)}{\sqrt{2\gamma_0(1/q)\cdot 2\gamma_1(1/q)}}\\
=&\frac{\mathrm{Cov}\left(Z_{ij}, Z_{sq}\right)-\mathrm{Cov}\left(Z_{ij}, Z_{(2s)1}\right)-\mathrm{Cov}\left(Z_{i(j+1)}, Z_{sq}\right)+\mathrm{Cov}\left(Z_{i(j+1)}, Z_{(2s)1}\right)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\\
=&\frac{-\gamma_1(1/q)+\gamma_c(2/q)-\gamma_0(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2\right] \\
=&\sqrt{2\left[2\gamma_0(1/q)\right]^2 \cdot 2\left[2\gamma_1(1/q)\right]^2} \cdot \left[\frac{-\gamma_0(1/q)+\gamma_c(2/q)-\gamma_1(1/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\right]^2 \\
=&2\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathrm{Corr}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s+1)1}\right) \\
=&\frac{\mathrm{Cov}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s+1)1}\right)}{\sqrt{2\gamma_0(1/q)\cdot 2\gamma_1(1/q)}} \\
=&\frac{\mathrm{Cov}\left(Z_{ij}, Z_{sq}\right)-\mathrm{Cov}\left(Z_{ij}, Z_{(2s+1)1}\right)-\mathrm{Cov}\left(Z_{i(j+1)}, Z_{sq}\right)+\mathrm{Cov}\left(Z_{i(j+1)}, Z_{(2s+1)1}\right)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}} \\
=&\frac{-\gamma_1(1/q)+\gamma_u(1/2q,1/2q)+\gamma_c(2/q)-\gamma_u(1/2q,3/2q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&2\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&2\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2 \\
&+2\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2.
\end{aligned}
$$

2) If $k = 0$ and $l = 1$

$$\begin{aligned}
&\text{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s)1}\right)\\
=&\frac{\text{Cov}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s)1}\right)}{\sqrt{2\gamma_0(1/q)\cdot 2\gamma_1(1/q)}}\\
=&\frac{\text{Cov}\left(Z_{ij}, Z_{sq}\right) - \text{Cov}\left(Z_{ij}, Z_{(2s)1}\right) - \text{Cov}\left(Z_{i(j+1)}, Z_{sq}\right) + \text{Cov}\left(Z_{i(j+1)}, Z_{(2s)1}\right)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\\
=&\frac{-2\gamma_c(2/q) + \gamma_0(1/q) + \gamma_c(3/q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}$$

and

$$\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2\right] = 2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2.$$

Similarly,

$$\begin{aligned}
&\text{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s+1)1}\right)\\
=&\frac{\text{Cov}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s+1)1}\right)}{\sqrt{2\gamma_0(1/q)\cdot 2\gamma_1(1/q)}}\\
=&\frac{\text{Cov}\left(Z_{ij}, Z_{sq}\right) - \text{Cov}\left(Z_{ij}, Z_{(2s+1)1}\right) - \text{Cov}\left(Z_{i(j+1)}, Z_{sq}\right) + \text{Cov}\left(Z_{i(j+1)}, Z_{(2s+1)1}\right)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}}\\
=&\frac{-\gamma_c(2/q) + \gamma_u(1/2q, 3/2q) + \gamma_c(3/q) - \gamma_u(1/2q, 5/2q)}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},
\end{aligned}$$

and

$$\begin{aligned}
&\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_c(2/q) - \gamma_c(3/q) - \gamma_u(1/2q, 3/2q) + \gamma_u(1/2q, 5/2q)\right]^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\text{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right]\\
=&2\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2\\
&+ 2\left[\gamma_c(2/q) - \gamma_c(3/q) - \gamma_u(1/2q, 3/2q) + \gamma_u(1/2q, 5/2q)\right]^2.
\end{aligned}$$

3) If $l \geq 2$

$$\operatorname{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s)1}\right) = \frac{-2\gamma_c[(l+1)/q] + \gamma_c(l/q) + \gamma_c[(l+2)/q]}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},$$

and

$$\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2\right] = 2\Big\{\gamma_c(l/q) - 2\gamma_c[(l+1)/q] + \gamma_c[(l+2)/q]\Big\}^2.$$

Similarly,

$$\begin{aligned}&\operatorname{Corr}\left(Z_{ij} - Z_{i(j+1)}, Z_{sq} - Z_{(2s+1)1}\right)\\=&\frac{-\gamma_c[(l+1)/q] + \gamma_u[1/2q, (2l+1)/2q] + \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+3)/2q]}{2\sqrt{\gamma_0(1/q)\gamma_1(1/q)}},\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right]\\=&2\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2.\end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{Cov}\left[\left(Z_{ij} - Z_{i(j+1)}\right)^2, \left(Z_{sq} - Z_{(2s)1}\right)^2 + \left(Z_{sq} - Z_{(2s+1)1}\right)^2\right]\\=&2\Big\{\gamma_c(l/q) - 2\gamma_c[(l+1)/q] + \gamma_c[(l+2)/q]\Big\}^2\\&+2\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2.\end{aligned}$$

k	# of Segments (i, s) Pairs
0	$\lvert A\rvert - 1$
1	$(\lvert A\rvert - 1) - 2 = \lvert A\rvert - 3$
$\vdots$	$\vdots$
k	$(\lvert A\rvert - 1) - (2^{k+1} - 2) = \lvert A\rvert - 2^{k+1} + 1$
$\vdots$	$\vdots$
$\lfloor m\rfloor$	$(\lvert A\rvert - 1) - (2^{\lfloor m\rfloor+1} - 2) = \lvert A\rvert - 2^{\lfloor m\rfloor+1} + 1$

Table 15: # of segments (i, s) pairs by k when i, s are FC and $i > s$

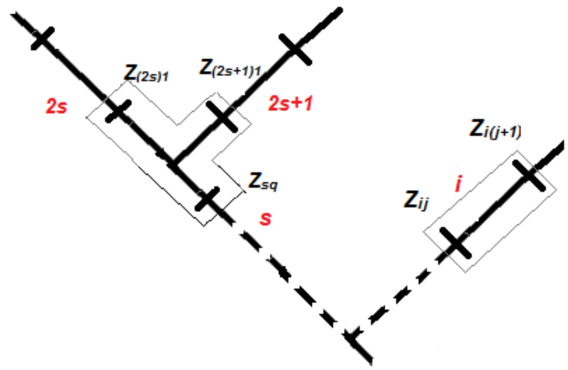
The cardinalities of segments (i, s) pairs by k under the scenario that segments i, s are FC and $i > s$

is tabulated in Table 15. Thus, the summation of $\mathrm{Cov}\Big[\sum_{j=1}^{q-1}\big(Z_{ij}-Z_{i(j+1)}\big)^2, \big(Z_{sq}-Z_{(2s)1}\big)^2 + \big(Z_{sq}-Z_{(2s+1)1}\big)^2\Big]$ under this scenario is

$$
\begin{aligned}
&(|A|-1)\cdot 2\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2\\
&+(|A|-1)\cdot 2\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2\\
&+(|A|-1)\cdot \mathrm{I}\{q\geq 3\}\cdot 2\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+(|A|-1)\cdot \mathrm{I}\{q\geq 3\}\cdot 2\left[\gamma_c(2/q)-\gamma_c(3/q)-\gamma_u(1/2q,3/2q)+\gamma_u(1/2q,5/2q)\right]^2\\
&+(|A|-1)\sum_{l=2}^{q-2}2\Big\{\gamma_c(l/q)-2\gamma_c[(l+1)/q]+\gamma_c[(l+2)/q]\Big\}^2\\
&+(|A|-1)\sum_{l=2}^{q-2}2\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2\\
&+\sum_{k=1}^{\lfloor m\rfloor}(|A|-2^{k+1}+1)\sum_{l=qk}^{q(k+1)-2}2\Big\{\gamma_c(l/q)-2\gamma_c[(l+1)/q]+\gamma_c[(l+2)/q]\Big\}^2\\
&+\sum_{k=1}^{\lfloor m\rfloor}(|A|-2^{k+1}+1)\sum_{l=qk}^{q(k+1)-2}2\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2\\
=&2(|A|-1)\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2\\
&+2(|A|-1)\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2\\
&+\mathrm{I}\{q\geq 3\}\cdot 2(|A|-1)\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2\\
&+\mathrm{I}\{q\geq 3\}\cdot 2(|A|-1)\left[\gamma_c(2/q)-\gamma_c(3/q)-\gamma_u(1/2q,3/2q)+\gamma_u(1/2q,5/2q)\right]^2\\
&+2(|A|-1)\sum_{l=2}^{q-2}\Big\{\gamma_c(l/q)-2\gamma_c[(l+1)/q]+\gamma_c[(l+2)/q]\Big\}^2\\
&+2(|A|-1)\sum_{l=2}^{q-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2\\
&+2\sum_{k=1}^{\lfloor m\rfloor}(|A|-2^{k+1}+1)\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c(l/q)-2\gamma_c[(l+1)/q]+\gamma_c[(l+2)/q]\Big\}^2\\
&+2\sum_{k=1}^{\lfloor m\rfloor}(|A|-2^{k+1}+1)\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c[(l+1)/q]-\gamma_c[(l+2)/q]-\gamma_u[1/2q,(2l+1)/2q]+\gamma_u[1/2q,(2l+3)/2q]\Big\}^2.
\end{aligned}
$$

III. Segments i,s are FU, and for (Z_{i1}, Z_{sq}), $\max\{a,b\} < m+1/q$

Assume for pair (Z_{i1}, Z_{sq}), $a=(k-1)+1/2q=(2kq-2q+1)/2q$ and $b=l-1/2q=(2lq-1)/2q$, $k,l\in\{1,...,\lceil m\rceil\}$. For fixed segments i,s such that for (Z_{i1}, Z_{sq}), $a=(2kq-2q+1)/2q$ and $b=(2lq-1)/2q$, let the distances-to-common-junction for (Z_{ij}, Z_{sq}) be $a=(2u+1)/2q$ and $b=(2lq-1)/2q$, $u\in\{q(k-1), q(k-1)+1, ..., qk-2\}$



We have

$$
\begin{aligned}
&\operatorname{Cov}\left(Z_{ij}-Z_{i(j+1)}, Z_{sq}-Z_{(2s)1}\right)\\
=&\operatorname{Cov}\left(Z_{ij}, Z_{sq}\right)-\operatorname{Cov}\left(Z_{ij}, Z_{(2s)1}\right)-\operatorname{Cov}\left(Z_{i(j+1)}, Z_{sq}\right)+\operatorname{Cov}\left(Z_{i(j+1)}, Z_{(2s)1}\right)\\
=&-\gamma_u(a\wedge b, a\vee b)+\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]\\
&+\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]-\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2\right]\\
=&2\Big\{\gamma_u(a\wedge b, a\vee b)-\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]\\
&-\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]+\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)]\Big\}^2.
\end{aligned}
$$

Since

$$
\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2\right]=\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s+1)1}\right)^2\right],
$$

we have

$$
\begin{aligned}
&\operatorname{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]\\
=&4\Big\{\gamma_u(a\wedge b, a\vee b)-\gamma_u[a\wedge(b+1/q), a\vee(b+1/q)]\\
&-\gamma_u[(a+1/q)\wedge b,(a+1/q)\vee b]\\
&+\gamma_u[(a+1/q)\wedge(b+1/q),(a+1/q)\vee(b+1/q)]\Big\}^2. \qquad \text{(a.1.3)}
\end{aligned}
$$

1) If $k \geq l+1$

When $k \geq l+1$, $b < b+1/q \leq a < a+1/q$, then by (a.1.3),

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&4\big[\gamma_u(b,a)-\gamma_u(b+1/q,a)-\gamma_u(b,a+1/q)+\gamma_u(b+1/q,a+1/q)\big]^2 \\
=&4\Big\{\gamma_u[(2lq-1)/2q,(2u+1)/2q]-\gamma_u[(2lq+1)/2q,(2u+1)/2q] \\
&\quad-\gamma_u[(2lq-1)/2q,(2u+3)/2q]+\gamma_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2.
\end{aligned}$$

2) If $k < l+1$, i.e., $k \leq l$

When $k \leq l$, $a < a+1/q \leq b < b+1/q$, then by (a.1.3),

$$\begin{aligned}
&\mathrm{Cov}\left[\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right] \\
=&4\big[\gamma_u(a,b)-\gamma_u(a,b+1/q)-\gamma_u(a+1/q,b)+\gamma_u(a+1/q,b+1/q)\big]^2 \\
=&4\Big\{\gamma_u[(2u+1)/2q,(2lq-1)/2q]-\gamma_u[(2u+1)/2q,(2lq+1)/2q] \\
&\quad-\gamma_u[(2u+3)/2q,(2lq-1)/2q]+\gamma_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2.
\end{aligned}$$

The cardinalities of segments (i,s) pairs by (k,l) under the scenario that segments i,s are FU is tabulated in Table 16.

$k \backslash l$	1	2	3	$\cdots$	l	$\cdots$	$\lceil m \rceil$
1	$(\lvert A\rvert-3)/2$	$(\lvert A\rvert-7)/2$	$(\lvert A\rvert-15)/2$	$\cdots$	$(\lvert A\rvert-2^{l+1}+1)/2$	$\cdots$	$(\lvert A\rvert-2^{\lceil m\rceil+1}+1)/2$
2	$\lvert A\rvert-3$	$\lvert A\rvert-7$	$\lvert A\rvert-15$	$\cdots$	$\lvert A\rvert-2^{l+1}+1$	$\cdots$	$\lvert A\rvert-2^{\lceil m\rceil+1}+1$
3	$\lvert A\rvert-7$	$2(\lvert A\rvert-7)$	$2(\lvert A\rvert-15)$	$\cdots$	$2(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2(\lvert A\rvert-2^{\lceil m\rceil+1}+1)$
4	$\lvert A\rvert-15$	$2(\lvert A\rvert-15)$	$2^2(\lvert A\rvert-15)$	$\cdots$	$2^2(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2^2(\lvert A\rvert-2^{\lceil m\rceil+1}+1)$
5	$\lvert A\rvert-31$	$2(\lvert A\rvert-31)$	$2^2(\lvert A\rvert-31)$	$\cdots$	$2^3(\lvert A\rvert-2^{l+1}+1)$	$\cdots$	$2^3(\lvert A\rvert-2^{\lceil m\rceil+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
k	$\lvert A\rvert-2^k+1$	$2(\lvert A\rvert-2^k+1)$	$2^2(\lvert A\rvert-2^k+1)$	$\cdots$	$2^{k\wedge(l+1)-2}(\lvert A\rvert-2^{k\vee(l+1)}+1)$	$\cdots$	$2^{k-2}(\lvert A\rvert-2^{\lceil m\rceil+1}+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\lceil m\rceil$	$\lvert A\rvert-2^{\lceil m\rceil}+1$	$2(\lvert A\rvert-2^{\lceil m\rceil}+1)$	$2^2(\lvert A\rvert-2^{\lceil m\rceil}+1)$	$\cdots$	$2^{l-1}(\lvert A\rvert-2^{\lceil m\rceil}+1)$	$\cdots$	$2^{\lceil m\rceil-2}(\lvert A\rvert-2^{\lceil m\rceil+1}+1)$

Table 16: # of segments (i,s) pairs by (k,l) when i,s are FU

Thus, the summation of $\mathrm{Cov}\Big[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2, \left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\Big]$ under this scenario is

$$\sum_{k=2}^{\lceil m\rceil}\sum_{l=1}^{k-1}2^{l-1}(|A|-2^k+1)\sum_{u=q(k-1)}^{qk-2}4\Big\{\gamma_u[(2lq-1)/2q,(2u+1)/2q]-\gamma_u[(2lq+1)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2lq-1)/2q,(2u+3)/2q]+\gamma_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2$$
$$+\sum_{l=1}^{\lceil m\rceil}\sum_{k=1}^{l}2^{k-2}(|A|-2^{l+1}+1)\sum_{u=q(k-1)}^{qk-2}4\Big\{\gamma_u[(2u+1)/2q,(2lq-1)/2q]-\gamma_u[(2u+1)/2q,(2lq+1)/2q]$$
$$-\gamma_u[(2u+3)/2q,(2lq-1)/2q]+\gamma_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2$$
$$=\sum_{k=2}^{\lceil m\rceil}\sum_{l=1}^{k-1}2^{l+1}(|A|-2^k+1)\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2lq-1)/2q,(2u+1)/2q]-\gamma_u[(2lq+1)/2q,(2u+1)/2q]$$
$$-\gamma_u[(2lq-1)/2q,(2u+3)/2q]+\gamma_u[(2lq+1)/2q,(2u+3)/2q]\Big\}^2$$
$$+\sum_{l=1}^{\lceil m\rceil}\sum_{k=1}^{l}2^{k}(|A|-2^{l+1}+1)\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2u+1)/2q,(2lq-1)/2q]-\gamma_u[(2u+1)/2q,(2lq+1)/2q]$$
$$-\gamma_u[(2u+3)/2q,(2lq-1)/2q]+\gamma_u[(2u+3)/2q,(2lq+1)/2q]\Big\}^2.$$

IV. Otherwise

In this scenario, random vectors $(Z_{i1},...,Z_{iq})$ and $(Z_{sq},Z_{(2s)1},Z_{(2s+1)1})$ are uncorrelated, thus

$$\mathrm{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]=0.$$

Summarizing I-IV, we have

$$|A|\mathrm{Cov}\left[\widehat{\gamma}_0(1/q),\widehat{\gamma}_1(1/q)\right]$$
$$=\frac{1}{4(q-1)(|A|-1)}\sum_{i=1}^{|A|}\sum_{s=1}^{(|A|-1)/2}\mathrm{Cov}\left[\sum_{j=1}^{q-1}\left(Z_{ij}-Z_{i(j+1)}\right)^2,\left(Z_{sq}-Z_{(2s)1}\right)^2+\left(Z_{sq}-Z_{(2s+1)1}\right)^2\right]$$
$$=\frac{1}{4(q-1)(|A|-1)}\Bigg\{2(|A|-1)\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2+\mathrm{I}\{q\geq 3\}\cdot 2(|A|-1)\left[\gamma_0(1/q)-2\gamma_c(2/q)+\gamma_c(3/q)\right]^2$$
$$+2(|A|-1)\sum_{l=3}^{q-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2$$
$$+\sum_{k=1}^{\lfloor m\rfloor}2(|A|-2^{k+1}+1)\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c\left[(l-1)/q\right]-2\gamma_c(l/q)+\gamma_c\left[(l+1)/q\right]\Big\}^2$$
$$+2(|A|-1)\left[\gamma_0(1/q)+\gamma_1(1/q)-\gamma_c(2/q)\right]^2$$
$$+2(|A|-1)\left[\gamma_1(1/q)-\gamma_c(2/q)-\gamma_u(1/2q,1/2q)+\gamma_u(1/2q,3/2q)\right]^2$$

$$
\begin{aligned}
&+ \mathrm{I}\{q \geq 3\} \cdot 2(|A|-1)\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ \mathrm{I}\{q \geq 3\} \cdot 2(|A|-1)\left[\gamma_c(2/q) - \gamma_c(3/q) - \gamma_u(1/2q, 3/2q) + \gamma_u(1/2q, 5/2q)\right]^2 \\
&+ 2(|A|-1)\sum_{l=2}^{q-2}\Big\{\gamma_c(l/q) - 2\gamma_c[(l+1)/q] + \gamma_c[(l+2)/q]\Big\}^2 \\
&+ 2(|A|-1)\sum_{l=2}^{q-2}\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2 \\
&+ 2\sum_{k=1}^{\lfloor m \rfloor}(|A| - 2^{k+1} + 1)\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c(l/q) - 2\gamma_c[(l+1)/q] + \gamma_c[(l+2)/q]\Big\}^2 \\
&+ 2\sum_{k=1}^{\lfloor m \rfloor}(|A| - 2^{k+1} + 1)\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2 \\
&+ \sum_{k=2}^{\lceil m \rceil}\sum_{l=1}^{k-1} 2^{l+1}(|A| - 2^k + 1)\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2lq-1)/2q, (2u+1)/2q] - \gamma_u[(2lq+1)/2q, (2u+1)/2q] \\
&\qquad - \gamma_u[(2lq-1)/2q, (2u+3)/2q] + \gamma_u[(2lq+1)/2q, (2u+3)/2q]\Big\}^2 \\
&+ \sum_{l=1}^{\lceil m \rceil}\sum_{k=1}^{l} 2^{k}(|A| - 2^{l+1} + 1)\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2u+1)/2q, (2lq-1)/2q] - \gamma_u[(2u+1)/2q, (2lq+1)/2q] \\
&\qquad - \gamma_u[(2u+3)/2q, (2lq-1)/2q] + \gamma_u[(2u+3)/2q, (2lq+1)/2q]\Big\}^2 \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{q-1}\left[\gamma_0(1/q) + \gamma_1(1/q) - \gamma_c(2/q)\right]^2 + \mathrm{I}\{q \geq 3\} \cdot \frac{1}{q-1}\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\
&+ \frac{1}{q-1}\sum_{l=3}^{q-1}\Big\{\gamma_c\left[(l-1)/q\right] - 2\gamma_c(l/q) + \gamma_c\left[(l+1)/q\right]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{k=1}^{\lfloor m \rfloor}\frac{|A| - 2^{k+1} + 1}{|A|-1}\sum_{l=qk+1}^{q(k+1)-1}\Big\{\gamma_c\left[(l-1)/q\right] - 2\gamma_c(l/q) + \gamma_c\left[(l+1)/q\right]\Big\}^2 \\
&+ \frac{1}{2(q-1)}\left[\gamma_1(1/q) - \gamma_c(2/q) - \gamma_u(1/2q, 1/2q) + \gamma_u(1/2q, 3/2q)\right]^2 \\
&+ \mathrm{I}\{q \geq 3\} \cdot \frac{1}{2(q-1)}\left[\gamma_c(2/q) - \gamma_c(3/q) - \gamma_u(1/2q, 3/2q) + \gamma_u(1/2q, 5/2q)\right]^2 \\
&+ \frac{1}{2(q-1)}\sum_{l=2}^{q-2}\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2 \\
&+ \frac{1}{2(q-1)}\sum_{k=1}^{\lfloor m \rfloor}\frac{|A| - 2^{k+1} + 1}{|A|-1}\sum_{l=qk}^{q(k+1)-2}\Big\{\gamma_c[(l+1)/q] - \gamma_c[(l+2)/q] - \gamma_u[1/2q, (2l+1)/2q] + \gamma_u[1/2q, (2l+3)/2q]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{k=2}^{\lceil m \rceil}\sum_{l=1}^{k-1}\frac{2^{l-1}(|A| - 2^k + 1)}{|A|-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2lq-1)/2q, (2u+1)/2q] - \gamma_u[(2lq+1)/2q, (2u+1)/2q] \\
&\qquad - \gamma_u[(2lq-1)/2q, (2u+3)/2q] + \gamma_u[(2lq+1)/2q, (2u+3)/2q]\Big\}^2 \\
&+ \frac{1}{q-1}\sum_{l=1}^{\lceil m \rceil}\sum_{k=1}^{l}\frac{2^{k-2}(|A| - 2^{l+1} + 1)}{|A|-1}\sum_{u=q(k-1)}^{qk-2}\Big\{\gamma_u[(2u+1)/2q, (2lq-1)/2q] - \gamma_u[(2u+1)/2q, (2lq+1)/2q] \\
&\qquad - \gamma_u[(2u+3)/2q, (2lq-1)/2q] + \gamma_u[(2u+3)/2q, (2lq+1)/2q]\Big\}^2.
\end{aligned}
$$

The facts that $l_{sn} \to \infty$ implies $|A| \to \infty$ and $\lim_{|A|\to\infty} \frac{|A|-a}{|A|-b} = 1$ for any $a, b \in \mathbb{R}$ complete the proof of Theorem 2.2.3. $\square$

A.2 Proof of Asymptotic Normality of $\underline{\widehat{\gamma}}^m$ (Theorem 2.3.1)

Proof. Let λ_1 and λ_2 be arbitrary real numbers (not both 0), and, recall that $|A| = 2^{l_{sn}} - 1$,

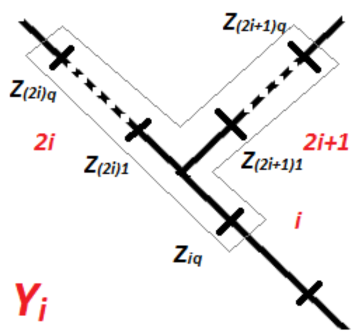
$$
\begin{aligned}
S_{l_{sn}} &= \sqrt{|A|}\Big\{\lambda_1 \left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] + \lambda_2 \left[\widehat{\gamma}_1(1/q) - \gamma_1(1/q)\right]\Big\} \\
&= \sqrt{|A|}\Bigg\{\lambda_1 \left[\frac{1}{2(q-1)|A|} \sum_{i=1}^{|A|}\sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2 - \gamma_0(1/q)\right] \\
&\qquad + \lambda_2 \left[\frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \left[\left(Z_{iq} - Z_{(2i)1}\right)^2 + \left(Z_{iq} - Z_{(2i+1)1}\right)^2\right] - \gamma_1(1/q)\right]\Bigg\} \\
&= \sqrt{|A|}\Bigg\{\frac{1}{2|A|} \sum_{i=1}^{|A|} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right] \\
&\qquad + \frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \left[\lambda_2 \left(Z_{iq} - Z_{(2i)1}\right)^2 + \lambda_2 \left(Z_{iq} - Z_{(2i+1)1}\right)^2 - 4\lambda_2\gamma_1(1/q)\right]\Bigg\} \\
&= \sqrt{|A|}\Bigg\{-\frac{1}{2|A|(|A|-1)} \sum_{i=1}^{|A|} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right] \\
&\qquad + \frac{1}{2(|A|-1)} \sum_{i=1}^{|A|} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right] \\
&\qquad + \frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \left[\lambda_2 \left(Z_{iq} - Z_{(2i)1}\right)^2 + \lambda_2 \left(Z_{iq} - Z_{(2i+1)1}\right)^2 - 4\lambda_2\gamma_1(1/q)\right]\Bigg\} \\
&= \sqrt{|A|}\Bigg\{-\frac{\lambda_1}{|A|-1} \left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] + \frac{1}{2(|A|-1)} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{1j} - Z_{1(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right] \\
&\qquad + \frac{1}{2(|A|-1)} \sum_{i=2}^{|A|} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{ij} - Z_{i(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right] \\
&\qquad + \frac{1}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \left[\lambda_2 \left(Z_{iq} - Z_{(2i)1}\right)^2 + \lambda_2 \left(Z_{iq} - Z_{(2i+1)1}\right)^2 - 4\lambda_2\gamma_1(1/q)\right]\Bigg\} \\
&= \frac{\sqrt{|A|}}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} \Bigg[\lambda_2 \left(Z_{iq} - Z_{(2i)1}\right)^2 + \lambda_2 \left(Z_{iq} - Z_{(2i+1)1}\right)^2 + \frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{(2i)j} - Z_{(2i)(j+1)}\right)^2 \\
&\qquad + \frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{(2i+1)j} - Z_{(2i+1)(j+1)}\right)^2 - 4\lambda_2\gamma_1(1/q) - 4\lambda_1\gamma_0(1/q)\Bigg] \\
&\quad - \frac{\lambda_1\sqrt{|A|}}{|A|-1} \left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] + \frac{\sqrt{|A|}}{2(|A|-1)} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{1j} - Z_{1(j+1)}\right)^2 - 2\lambda_1\gamma_0(1/q)\right]
\end{aligned}
$$

$$= \frac{\sqrt{|A|}}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) - \frac{\lambda_1 \sqrt{|A|}}{|A|-1} \left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right]$$
$$+ \frac{\sqrt{|A|}}{2(|A|-1)} \left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{1j} - Z_{1(j+1)}\right)^2 - 2\lambda_1 \gamma_0(1/q) \right], \qquad \text{(a.2.1)}$$

where $\mu_{\mathrm{Y}} = \mathrm{E}(\mathrm{Y}_i) = 4\lambda_2\gamma_1(1/q) + 4\lambda_1\gamma_0(1/q)$ and

$$\mathrm{Y}_i = \lambda_2 \left(Z_{i2} - Z_{(2i)1}\right)^2 + \lambda_2 \left(Z_{i2} - Z_{(2i+1)1}\right)^2 + \frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{(2i)j} - Z_{(2i)(j+1)}\right)^2$$
$$+ \frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{(2i+1)j} - Z_{(2i+1)(j+1)}\right)^2, i = 1, 2, ..., \frac{|A|-1}{2}. \qquad (7)$$

By definition, each Y_i involves $(2p+1)$ site around the node connecting segments i, $2i$ and $2i+1$. Hence in Figure 4, every node is labeled with associated Y_i.



It can be shown that

$$\mathrm{E}\left[\frac{\lambda_1}{q-1} \sum_{j=1}^{q-1} \left(Z_{1j} - Z_{1(j+1)}\right)^2 - 2\lambda_1 \gamma_0(1/q) \right] = \mathrm{E}\left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] = 0,$$

and by Theorem 2.2.1, we have

$$\mathrm{Var}\left[\widehat{\gamma}_0(1/q) - \gamma_0(1/q)\right] = \mathrm{Var}\left[\widehat{\gamma}_0(1/q)\right] < \infty.$$

Moreover, by the Cachy-Schwarz inequality

$$\begin{aligned}
&\operatorname{Var}\left[\frac{\lambda_1}{q-1}\sum_{j=1}^{q-1}\left(Z_{1j}-Z_{1(j+1)}\right)^2-2\lambda_1\gamma_0(1/q)\right]\\
=&\frac{\lambda_1^2}{(q-1)^2}\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}\operatorname{Cov}\left[\left(Z_{1j}-Z_{1(j+1)}\right)^2,\left(Z_{1k}-Z_{1(k+1)}\right)^2\right]\\
\leq&\frac{\lambda_1^2}{(q-1)^2}\cdot(q-1)^2\cdot\operatorname{Var}\left[\left(Z_{11}-Z_{12}\right)^2\right]=2\lambda_1^2\left[2\gamma_0(1/q)\right]^2<\infty.
\end{aligned}$$

Hence, by Chebyshev's inequality, it can be shown that, as $l_{sn} \to \infty$, which implies $|A| \to \infty$, the second and third term of (a.2.1) converge to 0 in probability. Thus by Slutsky's Theorem, if the first term of (a.2.1) converges in distribution, $S_{l_{sn}}$ converges to the same distribution.

Define $d(\mathrm{Y}_i, \mathrm{Y}_j) = d(Z_{iq}, Z_{jq})$. Y_i and Y_j are defined as flow-connected if any Z_{kl} formulating Y_i is flow-connected with any Z_{mn} formulating Y_j, otherwise Y_i and Y_j are defined as flow-unconnected. For example, in Figure 4, Y_2 and Y_4 are flow-connected, and Y_8 is flow-connected with Y_1; Y_2 is flow-unconnected with Y_3, and Y_6 and Y_7 are flow-unconnected.

Recall the number of levels of a stream network $l_{sn} = \log_2(|A| + 1)$. The number of levels of Y_i's $l_Y = l_{sn} - 1 = \log_2(|A| + 1) - 1$. For example, in Figure 4, the number of segments $|A| = 63$, $l_{sn} = \log_2(|A| + 1) = 6$ and $l_Y = 5$. In order to construct independently and identically distributed random variables, given the number of levels of Y_i's l_Y and m-$(2m{+}1/q)$-dependence, the construction of $\{X_{l_Y,i}\}$ sequence is as follows:

Step 1. Split Y_i's to either below and including level $\lceil l_Y/2\rceil$ or above and including level $\lceil l_Y/2\rceil + 1$.

Step 2. For each Y_i at level $\lceil l_Y/2\rceil - \lfloor m + 1/q\rfloor + 1$, i.e., the level that is $\lfloor m + 1/q\rfloor - 1$ levels below the top level of the bottom split, treat all Y_j's on the sub-stream with Y_i as the outlet and above (and including) level $\lceil l_Y/2\rceil + 1$ as one group. It can be shown that Y_i's at level $\lceil l_Y/2\rceil - \lfloor m + 1/q\rfloor + 1$ are $\mathrm{Y}_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q\rfloor}}$, $\mathrm{Y}_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q\rfloor}+1}$, ..., $\mathrm{Y}_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q\rfloor+1}-1}$.

Step 3. Define sets

$$Q_i = \{j : \mathrm{Y}_j \text{ above (and including) level } \lceil l_Y/2\rceil + 1, \text{ and FC with } \mathrm{Y}_i \text{ at level } \lceil l_Y/2\rceil - \lfloor m + 1/q\rfloor + 1\}.$$

Construct sequence $\{X_{l_Y,i}\}$ as

$$X_{l_Y,i} = \frac{1}{\sqrt{\left|Q_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor} + i - 1}\right|}} \sum_{j \in Q_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor} + i - 1}} (Y_j - \mu_Y),$$

for $i = 1, 2, ..., 2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor}$. It can be shown that, given l_Y,

$$\left|Q_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor} + i - 1}\right| = 2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor} - 1\right).$$

Thus

$$X_{l_Y,i} = \frac{1}{\sqrt{2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor} - 1\right)}} \sum_{j \in Q_{2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor} + i - 1}} (Y_j - \mu_Y),$$

for $i = 1, 2, ..., 2^{\lceil l_Y/2\rceil - \lfloor m+1/q \rfloor}$.

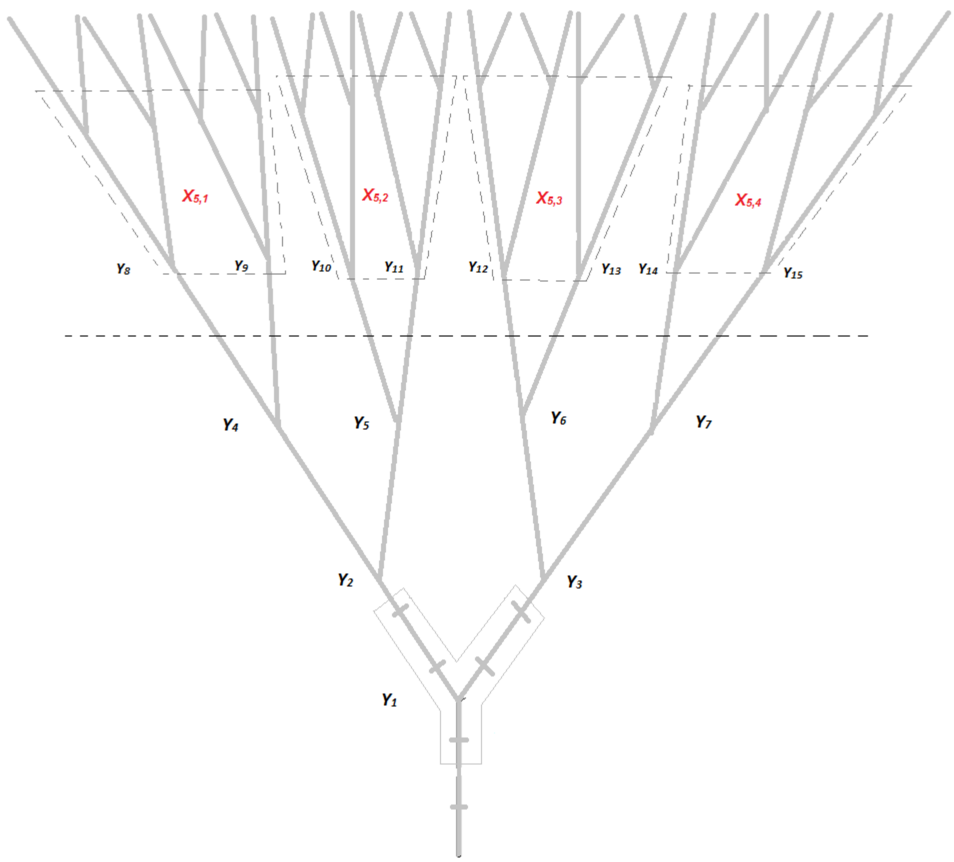


Figure 4: Y_i's in a 6-level rooted binary tree stream network

Figure 4 shows an example in a 6-level stream network, when $q = 2$, under 0.5-1.5-dependence. $l_Y = 5$ and the split of Y_i's is below level $\lceil l_Y/2 \rceil = 3$ and above level $\lceil l_Y/2 \rceil + 1 = 4$. The sequence of $\{\mathrm{X}_{5,i}\}$ has length of $2^{\lceil 5/2 \rceil - \lfloor 0.5+1/2 \rfloor} = 4$, and each $\mathrm{X}_{5,i}$ contains $2^{\lfloor 0.5+1/2 \rfloor} \left(2^{\lfloor 5/2 \rfloor} - 1\right) = 6$ Y_i's.

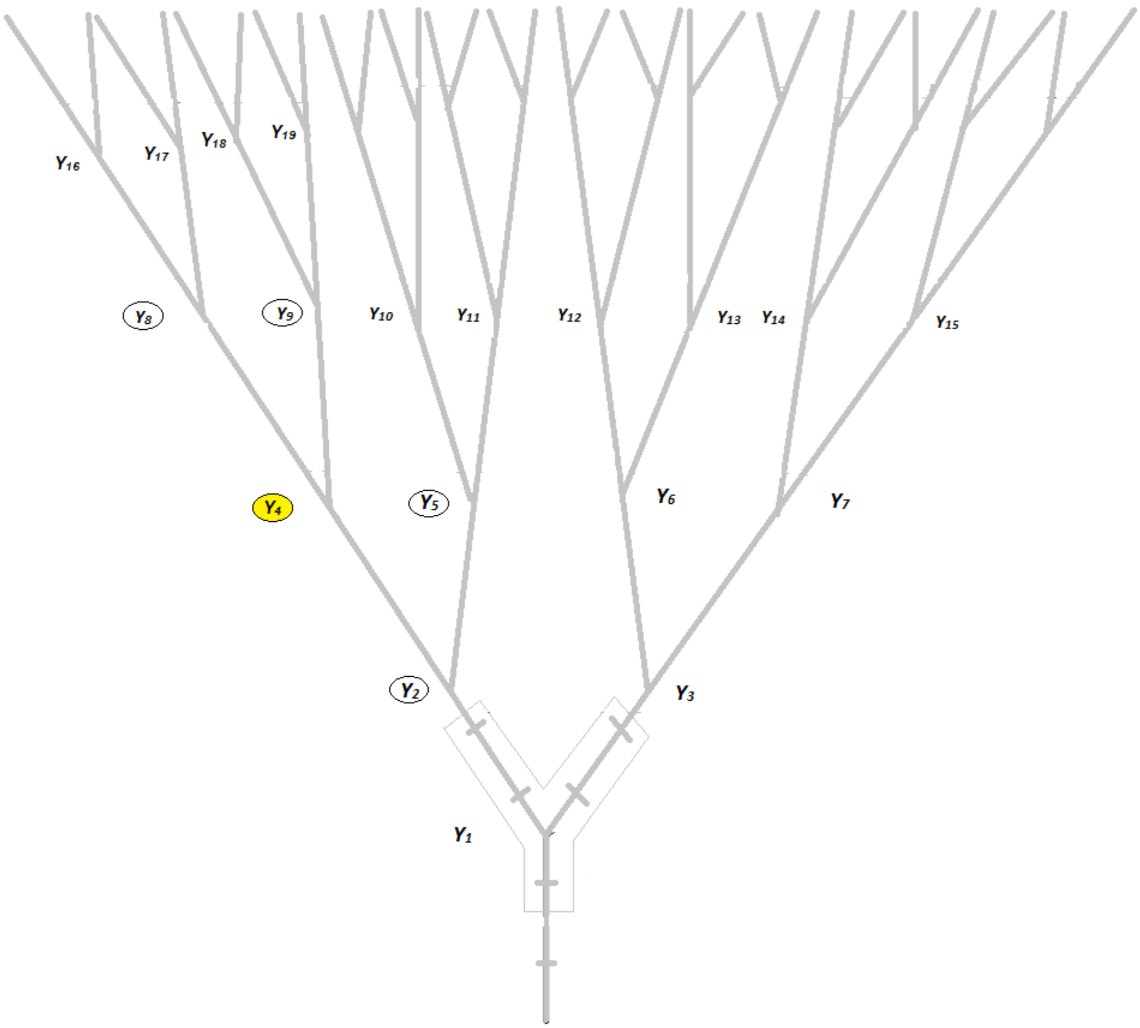


Figure 5: Y_i's dependent with Y_4 in a 6-level rooted binary tree stream network (circled)

$\mathrm{X}_{l_Y,i}$ has two nice properties:

P1. $\mathrm{E}(\mathrm{X}_{l_Y,i}) = 0$, which is trivial.

P2. $\mathrm{Var}(\mathrm{X}_{l_Y,i}) = \sigma^2_{\mathrm{X},l_Y} \to \sigma^2_{\mathrm{X}}$ as $l_Y \to \infty$, for some $\sigma^2_{\mathrm{X}} < \infty$.

Proof.

$$\mathrm{Var}(\mathrm{X}_{l_Y,i}) = \frac{1}{2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor}-1\right)} \cdot \sum_{j \in Q_{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}+i-1}} \sum_{k \in Q_{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}+i-1}} \mathrm{Cov}\left(\mathrm{Y}_j - \mu_\mathrm{Y}, \mathrm{Y}_k - \mu_\mathrm{Y}\right).$$

For a fixed j, the value of $\sum_k \mathrm{Cov}\left(\mathrm{Y}_j - \mu_\mathrm{Y}, \mathrm{Y}_k - \mu_\mathrm{Y}\right)$ depends on the location of Y_j. Since $\{Z_{ij}\}$ is m-$(2m{+}1/q)$-dependent, the dependence among $\left\{\mathrm{Y}_j\right\}$ is also within a limited range in FC or FU dimensions. Thus, the largest possible number of non-zero $\mathrm{Cov}\left(\mathrm{Y}_j - \mu_\mathrm{Y}, \mathrm{Y}_k - \mu_\mathrm{Y}\right)$ is finite and invariant of sufficiently large $|A|$. Define this number as N_Y.

For example, when $q = 2$ and $m = 0.5$, Y_4 in Figure 5 is dependent with Y_2 from downstream, with Y_8 and Y_9 from upstream, and with Y_5 from FU part. While for Y_1, it is only dependent with Y_2 and Y_3, 3 Y_j's including itself. Hence when $q = 2$ and $m = 0.5$, $N_Y = 5$.

For a fixed $\mathrm{X}_{l_Y,i}$, it contains $\lfloor l_Y/2 \rfloor$ levels of Y_j's, starting with $2^{\lfloor m+1/q \rfloor}$ Y_j's on first level, $2^{\lfloor m+1/q \rfloor+1}$ Y_j's on second level, till $2^{\lfloor l_Y/2 \rfloor + \lfloor m+1/q \rfloor - 1}$ Y_j's on top level. The Y_j's on the levels close to the top or bottom will each has less than N_Y dependent Y_k's. The cases of Y_j's are summarized in Table 17.

Case	Level within $\mathrm{X}_{l_Y,i}$	# of Y_j's	# of Dep. Y_k's Upstream	# of Dep. Y_k's Downstream
1	1	$2^{\lfloor m+1/q \rfloor}$	$2\left(2^{\lceil m+1/q \rceil}-1\right)$	0
2	2	$2^{\lfloor m+1/q \rfloor+1}$	$2\left(2^{\lceil m+1/q \rceil}-1\right)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lceil m+1/q \rceil$	$\lceil m+1/q \rceil$	$2^{\lfloor m+1/q \rfloor+\lceil m+1/q \rceil-1}$	$2\left(2^{\lceil m+1/q \rceil}-1\right)$	$\lceil m+1/q \rceil - 1$
$\lceil m+1/q \rceil+1$	$\lceil m+1/q \rceil+1$, ..., $\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil$	$2^{\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil + \lfloor m+1/q \rfloor} - 2^{\lfloor m+1/q \rfloor + \lceil m+1/q \rceil}$	$2\left(2^{\lceil m+1/q \rceil}-1\right)$	$\lceil m+1/q \rceil$
$\lceil m+1/q \rceil+2$	$\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil + 1$	$2^{\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil + \lfloor m+1/q \rfloor}$	$2\left(2^{\lceil m+1/q \rceil - 1}-1\right)$	$\lceil m+1/q \rceil$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2\lceil m+1/q \rceil$	$\lfloor l_Y/2 \rfloor - 1$	$2^{\lfloor l_Y/2 \rfloor + \lfloor m+1/q \rfloor - 2}$	2	$\lceil m+1/q \rceil$
$2\lceil m+1/q \rceil+1$	$\lfloor l_Y/2 \rfloor$	$2^{\lfloor l_Y/2 \rfloor + \lfloor m+1/q \rfloor - 1}$	0	$\lceil m+1/q \rceil$

Table 17: Summary of Y_j's in a fixed $\mathrm{X}_{l_Y,i}$

Since $\{Z_{ij}\}$ is second-order stationary, the value of $\sum_k \mathrm{Cov}\left(\mathrm{Y}_j - \mu_\mathrm{Y}, \mathrm{Y}_k - \mu_\mathrm{Y}\right)$ within each case of Y_j is constant. Define the values of $\sum_k \mathrm{Cov}\left(\mathrm{Y}_j - \mu_\mathrm{Y}, \mathrm{Y}_k - \mu_\mathrm{Y}\right)$ under Case i as τ_i, $i = 1, 2, ..., 2\lceil m+1/q \rceil + 1$. By gaussianity of $\{Z_{ij}\}$, $\mathrm{E}(Z_{ij}^4) < \infty$. Hence, for fixed λ_1 and λ_2, $\mathrm{E}(\mathrm{Y}_i^2) < \infty$, and $\sigma_\mathrm{Y}^2 = \mathrm{Var}(\mathrm{Y}_1) < \infty$, and τ_i's are all bounded for $i = 1, 2, ..., 2\lceil m+1/q \rceil + 1$.

Then

$$\begin{aligned}
\mathrm{Var}(\mathrm{X}_{l_Y,i}) =& \frac{1}{2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor}-1\right)} \cdot \\
& \Big[2^{\lfloor m+1/q \rfloor}\tau_1 + 2^{\lfloor m+1/q \rfloor+1}\tau_2 + \cdots + 2^{\lfloor m+1/q \rfloor + \lceil m+1/q \rceil - 1}\tau_{\lceil m+1/q \rceil} \\
& + \left(2^{\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil + \lfloor m+1/q \rfloor} - 2^{\lceil m+1/q \rceil + \lfloor m+1/q \rfloor}\right)\tau_{\lceil m+1/q \rceil+1} \\
& + 2^{\lfloor l_Y/2 \rfloor - \lceil m+1/q \rceil + \lfloor m+1/q \rfloor}\tau_{\lceil m+1/q \rceil+2} + \cdots \\
& + 2^{\lfloor l_Y/2 \rfloor + \lfloor m+1/q \rfloor - 2}\tau_{2\lceil m+1/q \rceil} + 2^{\lfloor l_Y/2 \rfloor + \lfloor m+1/q \rfloor - 1}\tau_{2\lceil m+1/q \rceil+1} \Big] \\
\to & \frac{1}{2^{\lceil m+1/q \rceil}}\tau_{\lceil m+1/q \rceil+1} + \frac{1}{2^{\lceil m+1/q \rceil}}\tau_{\lceil m+1/q \rceil+2} + \cdots + \frac{1}{2^2}\tau_{2\lceil m+1/q \rceil} + \frac{1}{2}\tau_{2\lceil m+1/q \rceil+1} \\
=& \frac{1}{2^{\lceil m+1/q \rceil+1}}\tau_{\lceil m+1/q \rceil+1} + \sum_{i=1}^{\lceil m+1/q \rceil+1} \frac{1}{2^{\lceil m+1/q \rceil - i+2}}\tau_{\lceil m+1/q \rceil+i} < \infty,
\end{aligned}$$

as $l_Y \to \infty$. Let σ_{X}^2 be the limit of $\mathrm{Var}(\mathrm{X}_{l_Y,i})$ completes the proof of P2 of $\mathrm{X}_{l_Y,i}$. □

Recall $l_Y = l_{sn} - 1 = \log_2(|A|+1) - 1$, thus $|A| = 2^{l_Y+1} - 1$. $l_Y \to \infty$ if and only if $|A| \to \infty$. Then

$$\begin{aligned}
& \frac{1}{\sqrt{(|A|-1)/2}} \sum_{i=1}^{(|A|-1)/2} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \\
=& \frac{1}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{l_Y}-1} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \\
=& \frac{1}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil}-1} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) + \frac{1}{\sqrt{2^{l_Y}-1}} \sum_{i=2^{\lceil l_Y/2 \rceil}}^{2^{l_Y}-1} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \\
=& \frac{1}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil}-1} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) + \frac{\sqrt{2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor}-1\right)}}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}} \mathrm{X}_{l_Y,i}. \qquad \text{(a.2.2)}
\end{aligned}$$

By the fact that at most N_Y Y_j's are dependent with a fixed Y_i (including Y_i itself), as well as

Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\operatorname{Var}\left[\frac{1}{\sqrt{2^{l_Y}-1}}\sum_{i=1}^{2^{\lceil l_Y/2\rceil}-1}(\mathrm{Y}_i-\mu_{\mathrm{Y}})\right]\\
=&\frac{1}{2^{l_Y}-1}\sum_{i=1}^{2^{\lceil l_Y/2\rceil}-1}\sum_{j=1}^{2^{\lceil l_Y/2\rceil}-1}\operatorname{Cov}\left(\mathrm{Y}_i-\mu_{\mathrm{Y}},\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\\
\leq&\frac{1}{2^{l_Y}-1}\sum_{i=1}^{2^{\lceil l_Y/2\rceil}-1}N_Y\sigma_{\mathrm{Y}}^2=\frac{N_Y(2^{\lceil l_Y/2\rceil}-1)}{2^{l_Y}-1}\sigma_{\mathrm{Y}}^2,
\end{aligned}$$

where $\sigma_{\mathrm{Y}}^2=\operatorname{Var}(\mathrm{Y}_1)$. By gaussianity of $\{Z_{ij}\}$, $\mathrm{E}(Z_{ij}^2)<\infty$ and $\mathrm{E}(Z_{ij}^4)<\infty$. Hence, for fixed λ_1 and λ_2, $\sigma_{\mathrm{Y}}^2=\operatorname{Var}(\mathrm{Y}_1)<\infty$, and the upper bound above converges to 0 as $l_Y\to\infty$. Besides,

$$\mathrm{E}\left[\frac{1}{\sqrt{2^{l_Y}-1}}\sum_{i=1}^{2^{\lceil l_Y/2\rceil}-1}(\mathrm{Y}_i-\mu_{\mathrm{Y}})\right]=\frac{1}{\sqrt{2^{l_Y}-1}}\sum_{i=1}^{2^{\lceil l_Y/2\rceil}-1}\mathrm{E}\left(\mathrm{Y}_i-\mu_{\mathrm{Y}}\right)=0.$$

Thus the first term of (a.2.2) converges to 0 in mean square. Berry-Esseen Theorem, Theorem 3.4.4 in Durrett (2013), is used to show the asympotic normality of the second term of (a.2.2).

Theorem A.2.1. *Let X_1, X_2, ... be i.i.ds with $\mathrm{E}(X_i)=0$, $\mathrm{E}(X_i^2)=\sigma^2$ and $\mathrm{E}(|X_i|^3)=\rho<\infty$. If $F_n(x)$ is the distribution of $(X_1+...+X_n)/\sigma\sqrt{n}$ and $\mathcal{N}(x)$ is the standard normal distribution, then*

$$\sup_{x\in\mathbb{R}}|F_n(x)-\mathcal{N}(x)|\leq\frac{3\rho}{\sigma^3\sqrt{n}}.$$

For sequence $\{\mathrm{X}_{l_Y,i}\}$, the following conditions, required by Berry-Esseen Theorem, can be shown:

1) $\mathrm{E}(\mathrm{X}_{l_Y,i})=0$.

$$\mathrm{E}(\mathrm{X}_{l_Y,i})=\frac{1}{\sqrt{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}}\sum_j\mathrm{E}\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)=0.$$

2) $\mathrm{E}(\mathrm{X}_{l_Y,i}^2)=\operatorname{Var}(\mathrm{X}_{l_Y,i})=\sigma_{\mathrm{X},l_Y}^2\geq\sigma_{\mathrm{Y}}^2$, and $\sigma_{\mathrm{Y}}^2>0$.

Proof.

$$\begin{aligned}\mathrm{Var}(\mathrm{X}_{l_Y,i}) &= \frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)} \sum_{j\in Q_{2^{\lceil l_Y/2\rceil-\lfloor m+1/q\rfloor}+i-1}} \sum_{k\in Q_{2^{\lceil l_Y/2\rceil-\lfloor m+1/q\rfloor}+i-1}} \mathrm{Cov}\left(\mathrm{Y}_j-\mu_{\mathrm{Y}},\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)\\ &\geq \frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)} \sum_{j\in Q_{2^{\lceil l_Y/2\rceil-\lfloor m+1/q\rfloor}+i-1}} \sigma_{\mathrm{Y}}^2\\ &= \frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}\cdot 2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)\sigma_{\mathrm{Y}}^2=\sigma_{\mathrm{Y}}^2.\end{aligned}$$

Moreover, by (7), $\sigma_{\mathrm{Y}}^2=\mathrm{Var}(\mathrm{Y}_i)=0$ if and only if Y_i is a constant, which is the case if and only if $\lambda_1=\lambda_2=0$. However, λ_1 and λ_2 are not both 0 by requirement. Hence $\sigma_{\mathrm{Y}}^2>0$. □

3) $\mathrm{E}(|\mathrm{X}_{l_Y,i}|^3)\leq\rho$ for some $\rho<\infty$.

Proof. By the fact that $|x|^3\leq 1+x^4$ for $x\in\mathrm{R}$, $\mathrm{E}(\mathrm{X}_{l_Y,i}^4)$ is bounded implies that $\mathrm{E}(|\mathrm{X}_{l_Y,i}|^3)$ is also bounded. Furthermore,

$$\mathrm{E}(\mathrm{X}_{l_Y,i}^4)=\mathrm{Var}(\mathrm{X}_{l_Y,i}^2)+\left[\mathrm{E}(\mathrm{X}_{l_Y,i}^2)\right]^2=\mathrm{Var}(\mathrm{X}_{l_Y,i}^2)+\left[\mathrm{Var}(\mathrm{X}_{l_Y,i})\right]^2. \tag{8}$$

Since at most N_Y Y_k's are dependent with a fixed Y_j (including Y_j itself), also by the Cauchy-Schwartz inequality,

$$\begin{aligned}\mathrm{Var}(\mathrm{X}_{l_Y,i}) &= \frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}\sum_j\sum_k \mathrm{Cov}\left(\mathrm{Y}_j-\mu_{\mathrm{Y}},\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)\\ &\leq \frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}\sum_j N_Y\sigma_{\mathrm{Y}}^2=N_Y\sigma_{\mathrm{Y}}^2.\end{aligned} \tag{9}$$

Furthermore

$$\begin{aligned}&\mathrm{Var}(\mathrm{X}_{l_Y,i}^2)\\ =&\mathrm{Var}\left[\frac{1}{\sqrt{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}}\sum_j\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\right]^2\\ =&\mathrm{Var}\left[\frac{1}{2^{\lfloor m+1/q\rfloor}\left(2^{\lfloor l_Y/2\rfloor}-1\right)}\sum_j\sum_k\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)\right]\\ =&\frac{1}{2^{\lfloor m+1/q\rfloor+1}\left(2^{\lfloor l_Y/2\rfloor}-1\right)^2}\sum_j\sum_k\sum_l\sum_m \mathrm{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right),\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right].\end{aligned} \tag{10}$$

For fixed j, k, by the Cauchy-Schwartz inequality,

$$\begin{aligned}&\operatorname{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right),\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right]\\ \leq&\sqrt{\operatorname{Var}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)\right]}\cdot\sqrt{\operatorname{Var}\left[\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right]},\end{aligned}\tag{11}$$

and

$$\begin{aligned}\operatorname{Var}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)\right]&\leq \mathrm{E}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)^2\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)^2\right]\\ &=\operatorname{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)^2,\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)^2\right]+\mathrm{E}\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)^2\mathrm{E}\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right)^2\\ &\leq \operatorname{Var}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)^2\right]+\sigma_{\mathrm{Y}}^4\\ &=\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4,\end{aligned}$$

where $\sigma_{\mathrm{Y}^2}^2=\operatorname{Var}\left[\left(\mathrm{Y}_1-\mu_{\mathrm{Y}}\right)^2\right]$. By gaussianity of $\{Z_{ij}\}$, $\mathrm{E}(Z_{ij}^8)<\infty$. Hence, for fixed λ_1 and λ_2, $\mathrm{E}(\mathrm{Y}_1^4)<\infty$, and $\sigma_{\mathrm{Y}^2}^2<\infty$. Thus by (11),

$$\operatorname{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right),\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right]\leq\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4<\infty.\tag{12}$$

For fixed j, k, $\operatorname{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right),\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right]\neq 0$ if and only if either one of the following conditions holds:

(c1). both Y_l and Y_m are dependent with either Y_i or Y_j

(c2). one of Y_l and Y_m is dependent with on either Y_i or Y_j, and the other is dependent with first one but is independent of Y_i and Y_j

The maximum number of dependent Y_l's with given Y_i and Y_j is $2N_Y$, assuming no overlapping of dependent Y_l's from each one. Thus the maximum number of $(\mathrm{Y}_l, \mathrm{Y}_m)$ pairs satisfies (c1) is $(2N_Y)^2 = 4N_Y^2$. Assume Y_l is dependent on either Y_i or Y_j, which has $2N_Y$ possible choices at most. Possible choices of Y_m that depends on Y_l is at most N_Y. Thus the maximum number of $(\mathrm{Y}_l, \mathrm{Y}_m)$ pairs satisfies (c2) is $2\times 2N_Y\times N_Y = 4N_Y^2$. Thus, by (10) and (12),

$$\begin{aligned}&\operatorname{Var}(\mathrm{X}_{l_Y,i}^2)\\ =&\frac{1}{2^{\lfloor m+1/q\rfloor+1}\left(2^{\lfloor l_Y/2\rfloor}-1\right)^2}\sum_j\sum_k\sum_l\sum_m\operatorname{Cov}\left[\left(\mathrm{Y}_j-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_k-\mu_{\mathrm{Y}}\right),\left(\mathrm{Y}_l-\mu_{\mathrm{Y}}\right)\left(\mathrm{Y}_m-\mu_{\mathrm{Y}}\right)\right]\\ \leq&\frac{1}{2^{\lfloor m+1/q\rfloor+1}\left(2^{\lfloor l_Y/2\rfloor}-1\right)^2}\sum_j\sum_k(4N_Y^2+4N_Y^2)(\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4)=8N_Y^2(\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4)<\infty.\end{aligned}\tag{13}$$

By (8), (9) and (13),

$$\begin{aligned}\mathrm{E}(|\mathrm{X}_{l_Y,i}|^3) &\leq 1+\mathrm{E}(\mathrm{X}_{l_Y,i}^4)\\ &= 1+\mathrm{Var}(\mathrm{X}_{l_Y,i}^2)+[\mathrm{Var}(\mathrm{X}_{l_Y,i})]^2\\ &\leq 1+8N_Y^2(\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4)+(N_Y\sigma_{\mathrm{Y}}^2)^2.\end{aligned}$$

Let $\rho = 1+8N_Y^2(\sigma_{\mathrm{Y}^2}^2+\sigma_{\mathrm{Y}}^4)+(N_Y\sigma_{\mathrm{Y}}^2)^2$. Then $\mathrm{E}(|\mathrm{X}_{l_Y,i}|^3)\leq\rho$ and $\rho<\infty$. □

Given l_Y, define the number of $\mathrm{X}_{l_Y,i}$'s as $n_{\mathrm{X}} = 2^{\lfloor l_Y/2\rfloor-\lfloor m+1/q\rfloor}$, so that $l_Y\to\infty$ is equivalent to $n_{\mathrm{X}}\to\infty$. $F_{l_Y,n_{\mathrm{X}}}$ as the distribution function of $(\mathrm{X}_{l_Y,1}+...+\mathrm{X}_{l_Y,n_{\mathrm{X}}})/\sigma_{\mathrm{X},l_Y}\sqrt{n_{\mathrm{X}}}$. With condition 1), 2) and 3), by Berry-Essen Theorem (Theorem A.2.1),

$$\sup_{x\in\mathrm{R}}|F_{l_Y,n_{\mathrm{X}}}(x)-\mathcal{N}(x)|\leq\frac{3\rho_{\mathrm{X},l_Y}}{\sigma_{\mathrm{X},l_Y}^3\sqrt{n_{\mathrm{X}}}}\leq\frac{3\rho}{\sigma_{\mathrm{Y}}^3\sqrt{n_{\mathrm{X}}}}.$$

Then, the distribution function of $(\mathrm{X}_{l_Y,1}+...+\mathrm{X}_{l_Y,n_{\mathrm{X}}})/\sigma_{\mathrm{X},l_Y}\sqrt{n_{\mathrm{X}}}$ converges uniformly to the distribution function of a standard normal as $l_Y\to\infty$, i.e.,

$$\frac{1}{\sigma_{\mathrm{X},l_Y}\sqrt{n_{\mathrm{X}}}}\sum_{i=1}^{n_{\mathrm{X}}}\mathrm{X}_{l_Y,i}\xrightarrow{d}N(0,1),$$

as $l_Y\to\infty$. By property of $\sigma_{\mathrm{X},l_Y}^2$, P2 and Slutsky's theorem

$$\frac{1}{\sigma_{\mathrm{X},l_Y}\sqrt{n_{\mathrm{X}}}}\sum_{i=1}^{n_{\mathrm{X}}}\mathrm{X}_{l_Y,i}\cdot\sqrt{\frac{\sigma_{\mathrm{X},l_Y}^2}{\sigma_{\mathrm{X}}^2}}\xrightarrow{d}N(0,1),$$

as $l_Y\to\infty$. Thus,

$$\frac{1}{\sqrt{n_{\mathrm{X}}}}\sum_i\mathrm{X}_{l_Y,i}=\frac{1}{\sqrt{2^{\lfloor l_Y/2\rfloor-\lfloor m+1/q\rfloor}}}\sum_i\mathrm{X}_{l_Y,i}\xrightarrow{d}N(0,\sigma_{\mathrm{X}}^2),\tag{14}$$

as $l_Y \to \infty$. Thus, by Slutsky's Theorem, the second term of (a.2.2)

$$\begin{aligned}
&\frac{\sqrt{2^{\lfloor m+1/q \rfloor}\left(2^{\lfloor l_Y/2 \rfloor}-1\right)}}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}} \mathrm{X}_{l_Y,i} \\
=&\frac{\sqrt{2^{\lfloor l_Y/2 \rfloor}-1}\cdot\sqrt{2^{\lceil l_Y/2 \rceil}}}{\sqrt{2^{l_Y}-1}} \cdot \frac{1}{\sqrt{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil - \lfloor m+1/q \rfloor}} \mathrm{X}_{l_Y,i} \xrightarrow{d} N(0,\sigma_{\mathrm{X}}^2),
\end{aligned}$$

as $l_Y \to \infty$. Then, again by Slutsky's Theorem, (a.2.2)

$$\begin{aligned}
&\frac{1}{\sqrt{(|A|-1)/2}} \sum_{i=1}^{(|A|-1)/2} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \\
=&\frac{1}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil}-1} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) + \frac{\sqrt{2^{\lfloor l_Y/2 \rfloor+1}-2}}{\sqrt{2^{l_Y}-1}} \sum_{i=1}^{2^{\lceil l_Y/2 \rceil -1}} \mathrm{X}_{l_Y,i} \xrightarrow{d} N(0,\sigma_{\mathrm{X}}^2),
\end{aligned}$$

as $l_Y \to \infty$, i.e., as $l_{sn} \to \infty$. Hence, the first term of (a.2.1)

$$\begin{aligned}
&\frac{\sqrt{|A|}}{2(|A|-1)} \sum_{i=1}^{(|A|-1)/2} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \\
=&\frac{\sqrt{|A|}\cdot\sqrt{(|A|-1)/2}}{2(|A|-1)} \cdot \frac{1}{\sqrt{(|A|-1)/2}} \sum_{i=1}^{(|A|-1)/2} (\mathrm{Y}_i - \mu_{\mathrm{Y}}) \xrightarrow{d} N(0,\sigma_{\mathrm{X}}^2/8),
\end{aligned}$$

as $l_{sn} \to \infty$. $S_{l_{sn}}$ converges to the same distribution. Since λ_1 and λ_2 are arbitrary, by the Cramér-Wold device, the random vector

$$\sqrt{|A|}\begin{bmatrix} \widehat{\gamma}_0(1/q) - \gamma_0(1/q) \\ \widehat{\gamma}_1(1/q) - \gamma_1(1/q) \end{bmatrix} \xrightarrow{d} N\left(\underline{0}, \boldsymbol{\Sigma}^m\right),$$

as $l_{sn} \to \infty$ for some $\boldsymbol{\Sigma}^m$. The elements of $\boldsymbol{\Sigma}^m$ are identified by Theorem 2.2.1–2.2.3. □

A.3 Proof of Theorem 3.3.1

Proof. The pointwise convergence of two convariance functions will be proved separately.

1. Pointwise convergence of $C_{fc}^m(h)$

   For fixed $h \geq 0$,

$$\begin{aligned}
C_{fc}^m(h) &= \int_{-\infty}^{-h} g^m(-x)g^m(-x-h)\,dx \\
&= \int_{-\infty}^{-h} g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(0 \leq -x \leq m+1/q) \cap (0 \leq -x-h \leq m+1/q)\Big\}\,dx \\
&= \int_{-\infty}^{-h} g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(-m-1/q \leq x \leq 0) \cap (-h-m-1/q \leq x \leq -h)\Big\}\,dx \\
&= \int_{-\infty}^{\infty} g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(-m-1/q \leq x \leq 0) \cap (-h-m-1/q \leq x \leq -h)\Big\}\,dx.
\end{aligned}$$

   By unilateral property of $g(\cdot)$,

$$\begin{aligned}
C_{fc}(h) &= \int_{-\infty}^{-h} g(-x)g(-x-h)\,dx \\
&= \int_{-\infty}^{-h} g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(0 \leq -x) \cap (0 \leq -x-h)\Big\}\,dx \\
&= \int_{-\infty}^{-h} g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(x \leq 0) \cap (x \leq -h)\Big\}\,dx \\
&= \int_{-\infty}^{\infty} g(-x)g(-x-h)\cdot \mathrm{I}\{x \leq -h\}\,dx.
\end{aligned}$$

   Define $f_c^m(x) = g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(-m-1/q \leq x \leq 0) \cap (-h-m-1/q \leq x \leq -h)\Big\}$ and $f_c(x) = g(-x)g(-x-h)\cdot \mathrm{I}\{x \leq -h\}$. Then

$$\begin{aligned}
& f_c^{m+1/q}(x) - f_c^m(x) \\
=& g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(-m-2/q \leq x \leq 0) \cap (-h-m-2/q \leq x \leq -h)\Big\} \\
& - g(-x)g(-x-h)\cdot \mathrm{I}\Big\{(-m-1/q \leq x \leq 0) \cap (-h-m-1/q \leq x \leq -h)\Big\} \\
=& \begin{cases} 0 & \text{if } m < h-2/q, \\ g(-x)g(-x-h)\cdot \mathrm{I}\{-m-2/q \leq x \leq -h\} & \text{if } h-2/q \leq m < h-1/q, \\ g(-x)g(-x-h)\cdot \mathrm{I}\{-m-2/q \leq x < -m-1/q\} & \text{if } m \geq h-1/q. \end{cases}
\end{aligned}$$

By non-negative property of $g(\cdot)$, $f_c^{m+1/q}(x) - f_c^m(x) \geq 0$, or $f_c^{m+1/q}(x) \geq f_c^m(x) \geq 0$ for any $x \in \mathbb{R}$. When $m > h - 1/q$, again by non-negative property of $g(\cdot)$,

$$\begin{aligned} f_c(x) - f_c^m(x) &= g(-x)g(-x-h) \cdot \Big(\mathrm{I}\{x \leq -h\} - \mathrm{I}\{-m - 1/q \leq x \leq -h\}\Big) \\ &= g(-x)g(-x-h) \cdot \mathrm{I}\{x < -m - 1/q\} \geq 0. \end{aligned}$$

By the fact that $g(\cdot)$ is non-negative and $g \downarrow 0$ (since square-integrable), the difference above implies

$$\begin{aligned} \sup_x \Big| f_c(x) - f_c^m(x) \Big| &= \sup_x \Big[g(-x)g(-x-h) \cdot \mathrm{I}\{x < -m - 1/q\} \Big] \\ &= g(m + 1/q)g(m + 1/q - h) \to 0 \end{aligned}$$

as $m \to \infty$, which implies $f^m \to f$ pointwisely as $m \to \infty$. Then, by Monotone Convergence Theorem, $f^{m+1/q} \geq f^m \geq 0$, $f^m \to f$ pointwisely as $m \to \infty$ and the boundedness of $g(\cdot)$ implies that

$$C_{fc}^m(h) = \int_{-\infty}^{\infty} f^m(x)\, dx \to \int_{-\infty}^{\infty} f(x)\, dx = C_{fc}(h) \text{ as } m \to \infty.$$

Since $h \geq 0$ is arbitrary, $C_{fc}^m(h)$ converges to $C_{fc}(h)$ pointwisely.

2. Pointwise convergence of $C_{fu}^m(a, b)$

For fixed $b \geq a \geq 0$,

$$\begin{aligned} C_{fu}^m(a,b) &= \int_{-\infty}^{-b} g^m(-x)g^m(-x-(b-a))\, dx \\ &= \int_{-\infty}^{-b} g(-x)g(-x-(b-a)) \cdot \\ &\qquad \mathrm{I}\Big\{(0 \leq -x \leq m + 1/q) \cap (0 \leq -x - (b-a) \leq m + 1/q)\Big\}\, dx \\ &= \int_{-\infty}^{-b} g(-x)g(-x-(b-a)) \cdot \\ &\qquad \mathrm{I}\Big\{(-m - 1/q \leq x \leq 0) \cap (-b + a - m - 1/q \leq x \leq -b + a)\Big\}\, dx \end{aligned}$$

$$= \int_{-\infty}^{\infty} g(-x)g(-x-(b-a))\cdot$$
$$\mathrm{I}\Big\{(-m-1/q \le x \le 0) \cap (-b+a-m-1/q \le x \le -b+a) \cap (x \le -b)\Big\}\, dx.$$

By unilateral property of $g(\cdot)$,

$$\begin{aligned}
C_{fu}(a,b) &= \int_{-\infty}^{-b} g(-x)g(-x-(b-a))\, dx \\
&= \int_{-\infty}^{-b} g(-x)g(-x-(b-a)) \cdot \mathrm{I}\Big\{(0 \le -x) \cap (0 \le -x-(b-a))\Big\}\, dx \\
&= \int_{-\infty}^{-b} g(-x)g(-x-(b-a)) \cdot \mathrm{I}\Big\{(x \le 0) \cap (x \le -b+a)\Big\}\, dx \\
&= \int_{-\infty}^{\infty} g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{x \le -b\}\, dx.
\end{aligned}$$

Define

$$\begin{aligned}
f_u^m(x) = {} & g(-x)g(-x-(b-a))\cdot \\
& \mathrm{I}\Big\{(-m-1/q \le x \le 0) \cap (-b+a-m-1/q \le x \le -b+a) \cap (x \le -b)\Big\} \\
= {} & g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{-m-1/q \le x \le -b\}, \\
f_u(x) = {} & g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{x \le -b\}.
\end{aligned}$$

Then

$$\begin{aligned}
& f_u^{m+1/q}(x) - f_u^m(x) \\
= & g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{-m-2/q \le x \le -b\} \\
& - g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{-m-1/q \le x \le -b\} \\
= & \begin{cases} 0 & \text{if } m < b-2/q, \\ g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{-m-2/q \le x \le -b\} & \text{if } b-2/q \le m < b-1/q, \\ g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{-m-2/q \le x < -m-1/q\} & \text{if } m \ge b-1/q. \end{cases}
\end{aligned}$$

By non-negative property of $g(\cdot)$, $f_u^{m+1/q}(x) - f_u^m(x) \ge 0$, or $f_u^{m+1/q}(x) \ge f_u^m(x) \ge 0$ for any

$x \in \mathbb{R}$. When $m > b - 1/q$, again by non-negative property of $g(\cdot)$,

$$\begin{aligned} f_u(x) - f_u^m(x) &= g(-x)g(-x-(b-a)) \cdot \Big(\mathrm{I}\{x \le -b\} - \mathrm{I}\{-m-1/q \le x \le -b\} \Big) \\ &= g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{x < -m-1/q\} \ge 0. \end{aligned}$$

By the fact that $g(\cdot)$ is non-negative and $g \downarrow 0$ (since square-integrable), the difference above implies

$$\begin{aligned} \sup_x \Big| f_u(x) - f_u^m(x) \Big| &= \sup_x \Big[g(-x)g(-x-(b-a)) \cdot \mathrm{I}\{x < -m-1/q\} \Big] \\ &= g(m+1/q)g(m+1/q-(b-a)) \to 0 \end{aligned}$$

as $m \to \infty$, which implies $f_u^m \to f_u$ pointwisely as $m \to \infty$. Then, by Monotone Convergence Theorem, $f_u^{m+1/q} \ge f_u^m \ge 0$, $f^m \to f$ pointwisely as $m \to \infty$ and the boundedness of $g(\cdot)$ implies that

$$C_{fu}^m(a,b) = \int_{-\infty}^{\infty} f_u^m(x)\, dx \to \int_{-\infty}^{\infty} f_u(x)\, dx = C_{fu}(a,b) \text{ as } m \to \infty.$$

Since $b \ge a \ge 0$ are arbitrary, $C_{fu}^m(a,b)$ converges to $C_{fu}(a,b)$ pointwisely.

□

A.4 Proof of Theorem 3.4.1

Proof. This proof will elaborate $\sigma_{22}^m \to \sigma_{22} < \infty$ as $m \to \infty$. $\sigma_{11}^m \to \sigma_{11} < \infty$ and $\sigma_{12}^m \to \sigma_{12} < \infty$ as $m \to \infty$ can be proved in similar fashion.

By Theorem 3.2.1,

$$\begin{aligned} \sigma_{22}^{m*} =& \frac{1}{2}\left[2\gamma_1^m(1/q)\right]^2 + \frac{1}{2}\left[2\gamma_1^m(1/q) - \gamma_u^m(1/2q, 1/2q)\right]^2 + \mathrm{I}\{q=2\} \cdot \mathrm{I}\Big\{\lfloor m+1/q \rfloor \ge 1\Big\} \cdot \left[\gamma_0^m(1/q) - 2\gamma_c^m(2/q) + \gamma_c^m(3/q)\right]^2 \\ &+ \mathrm{I}\{q \ge 3\} \cdot \mathrm{I}\Big\{\lfloor m+1/q \rfloor \ge 1\Big\} \cdot \left[\gamma_c^m(1-1/q) - 2\gamma_c^m(1) + \gamma_c^m(1+1/q)\right]^2 \\ &+ \sum_{k=2}^{\infty} \left[\gamma_c^m(k-1/q) - 2\gamma_c^m(k) + \gamma_c^m(k+1/q)\right]^2 \\ &+ \sum_{k=1}^{\infty} \left[\gamma_c^m(k) - \gamma_c^m(k+1/q) - \gamma_u^m(1/2q, k-1/2q) + \gamma_u^m(1/2q, k+1/2q)\right]^2 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}2^{k-1}[\gamma_u^m(k-1/2q,k-1/2q)-2\gamma_u^m(k-1/2q,k+1/2q)+\gamma_u^m(k+1/2q,k+1/2q)]^2\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}2^l\big[\gamma_u^m(l-1/2q,k-1/2q)-\gamma_u^m(l+1/2q,k-1/2q)\\
&\qquad\qquad-\gamma_u^m(l-1/2q,k+1/2q)+\gamma_u^m(l+1/2q,k+1/2q)\big]^2, \qquad \text{(a.4.1)}
\end{aligned}
$$

and $\sigma_{22}^{m*}=\sigma_{22}^m$. By Theorem 3.3.1, $C_{fc}^m(h)$, $C_{fu}^m(a,b)$ converges to $C_{fc}(h)$, $C_{fu}(a,b)$ pointwisely as $m\to\infty$. Then, as $m\to\infty$,

$$
\begin{aligned}
\gamma_0^m(1/q)&=C_{fc}^m(0)-C_{fc}^m(1/q)\to C_{fc}(0)-C_{fc}(1/q)=\gamma_0(1/q),\\
\gamma_1^m(1/q)&=C_{fc}^m(0)-C_{fc}^m(1/q)\to C_{fc}(0)-C_{fc}(1/q)=\gamma_1(1/q),\\
\gamma_c^m(h)&=C_{fc}^m(0)-C_{fc}^m(h)\to C_{fc}(0)-C_{fc}(h)=\gamma_c(h).
\end{aligned}
$$

Then, as $m\to\infty$, the first four terms of (a.4.1) converge to the first four terms of σ_{22}. The proof of Theorem 3.3.1 with Monotone Convergence Theorem implies that

$$
C_{fc}^m(h)\le C_{fc}(h),\ C_{fu}^m(a,b)\le C_{fu}(a,b). \qquad (15)
$$

By (15) and the Cauchy-Schwartz inequality, the summand of the fifth term of (a.4.1)

$$
\begin{aligned}
&[\gamma_c^m(k-1/q)-2\gamma_c^m(k)+\gamma_c^m(k+1/q)]^2\\
=&\left[-C_{fc}^m(k-1/q)+2C_{fc}^m(k)-C_{fc}^m(k+1/q)\right]^2\\
\le&3\Big\{[C_{fc}^m(k-1/q)]^2+[2C_{fc}^m(k)]^2+[C_{fc}^m(k+1/q)]^2\Big\}\\
\le&3\Big\{[C_{fc}(k-1/q)]^2+[2C_{fc}(k)]^2+[C_{fc}(k+1/q)]^2\Big\}. \qquad (16)
\end{aligned}
$$

The absolute summability of $C_{fc}(\cdot)$ implies $C_{fc}(h)\to 0$ as $h\to\infty$. Then there exist $N>0$ such that, for any $k>N$, $\left|C_{fc}(k-1/q)\right|<1$ or $[C_{fc}(k-1/q)]^2<\left|C_{fc}(k-1/q)\right|$. Then

$$
\sum_{k=2}^{\infty}3\Big\{[C_{fc}(k-1/q)]^2+[2C_{fc}(k)]^2+[C_{fc}(k+1/q)]^2\Big\}
$$

$$
\begin{aligned}
=&\sum_{k=2}^{N} 3\Big\{[C_{fc}(k-1/q)]^2+4[C_{fc}(k)]^2+[C_{fc}(k+1/q)]^2\Big\}\\
&+\sum_{k=N+1}^{\infty} 3\Big\{[C_{fc}(k-1/q)]^2+4[C_{fc}(k)]^2+[C_{fc}(k+1/q)]^2\Big\}\\
\le&3\sum_{k=2}^{N}[C_{fc}(k-1/q)]^2+12\sum_{k=2}^{N}[C_{fc}(k)]^2+3\sum_{k=2}^{N}[C_{fc}(k+1/q)]^2\\
&+3\sum_{k=N+1}^{\infty}\big|C_{fc}(k-1/q)\big|+12\sum_{k=N+1}^{\infty}\big|C_{fc}(k)\big|+3\sum_{k=N+1}^{\infty}\big|C_{fc}(k+1/q)\big|\\
\le&3\sum_{k=2}^{N}[C_{fc}(2-1/q)]^2+12\sum_{k=2}^{N}[C_{fc}(2)]^2+3\sum_{k=2}^{N}[C_{fc}(2+1/q)]^2+18\sum_{i=0}^{\infty}\big|C_{fc}(i/q)\big|<\infty. \qquad (17)
\end{aligned}
$$

By Dominated Convergence Theorem for Sums, (16) and (17) and pointwise convergence of $C_{fc}^{m}(h)$ to $C_{fc}(h)$ implies that, as $m\to\infty$, the fifth term of (a.4.1)

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\left[\gamma_c^m(k-1/q)-2\gamma_c^m(k)+\gamma_c^m(k+1/q)\right]^2\\
\to&\sum_{k=2}^{\infty}\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2<\infty. \qquad (18)
\end{aligned}
$$

By (15) and the Cauchy-Schwartz inequality, the summand of the seventh term of (a.4.1)

$$
\begin{aligned}
&2^{k-1}\Big[\gamma_u^m(k-1/2q,k-1/2q)-2\gamma_u^m(k-1/2q,k+1/2q)+\gamma_u^m(k+1/2q,k+1/2q)\Big]^2\\
=&2^{k-1}\Big[-C_{fu}^m(k-1/2q,k-1/2q)+2C_{fu}^m(k-1/2q,k+1/2q)-C_{fu}^m(k+1/2q,k+1/2q)\Big]^2\\
\le&3\Big\{2^{k-1}\left[C_{fu}^m(k-1/2q,k-1/2q)\right]^2+2^{k+1}\left[C_{fu}^m(k-1/2q,k+1/2q)\right]^2\\
&+2^{k-1}\left[C_{fu}^m(k+1/2q,k+1/2q)\right]^2\Big\}. \qquad (19)
\end{aligned}
$$

The absolute summability of $C_{fu}(\cdot)$ implies that $2^{i/2}\big|C_{fu}(1/2q+j/q,1/2q+i/q)\big|\to 0$ as $i\to\infty$. Then there exist $N>0$ such that, for any $i>N$, $2^{i/2}\big|C_{fu}(1/2q+j/q,1/2q+i/q)\big|<1$ or $2^{i}\left[C_{fu}(1/2q+j/q,1/2q+i/q)\right]^2<2^{i/2}\big|C_{fu}(1/2q+j/q,1/2q+i/q)\big|$. Then by the fact that $2^{i/2}\big|C_{fu}(1/2q+j/q,1/2q+i/q)\big|<\infty$ for any $i\in\mathbb{N}_+$,

$$\sum_{k=1}^{\infty} 3\Big\{2^{k-1}\left[C_{fu}(k-1/2q,k-1/2q)\right]^2 + 2^{k+1}\left[C_{fu}(k-1/2q,k+1/2q)\right]^2 + 2^{k-1}\left[C_{fu}(k+1/2q,k+1/2q)\right]^2\Big\}$$

$$=\sum_{k=1}^{N} 3\Big\{2^{k-1}\left[C_{fu}(k-1/2q,k-1/2q)\right]^2 + 2^{k+1}\left[C_{fu}(k-1/2q,k+1/2q)\right]^2 + 2^{k-1}\left[C_{fu}(k+1/2q,k+1/2q)\right]^2\Big\}$$

$$+\sum_{k=N+1}^{\infty} 3\Big\{2^{k-1}\left[C_{fu}(k-1/2q,k-1/2q)\right]^2 + 2^{k+1}\left[C_{fu}(k-1/2q,k+1/2q)\right]^2 + 2^{k-1}\left[C_{fu}(k+1/2q,k+1/2q)\right]^2\Big\}$$

$$\le 3\sum_{k=1}^{N} 2^{k-1}\left[C_{fu}(k-1/2q,k-1/2q)\right]^2 + 3\sum_{k=1}^{N} 2^{k+1}\left[C_{fu}(k-1/2q,k+1/2q)\right]^2$$

$$+3\sum_{k=1}^{N} 2^{k-1}\left[C_{fu}(k+1/2q,k+1/2q)\right]^2$$

$$+3\sum_{k=N+1}^{\infty} 2^{(k-1)/2}\big|C_{fu}(k-1/2q,k-1/2q)\big| + 3\sum_{k=N+1}^{\infty} 2^{(k+1)/2}\big|C_{fu}(k-1/2q,k+1/2q)\big|$$

$$+3\sum_{k=N+1}^{\infty} 2^{(k-1)/2}\big|C_{fu}(k+1/2q,k+1/2q)\big|$$

$$\le 3\sum_{k=1}^{N} 2^{k-1}\left[C_{fu}(k-1/2q,k-1/2q)\right]^2 + 3\sum_{k=1}^{N} 2^{k+1}\left[C_{fu}(k-1/2q,k+1/2q)\right]^2$$

$$+3\sum_{k=1}^{N} 2^{k-1}\left[C_{fu}(k+1/2q,k+1/2q)\right]^2$$

$$+6\sqrt{2}\sum_{i=0}^{\infty}\sum_{j=0}^{i} 2^{j/2}\big|C_{fu}(j/q+1/2q,i/q+1/2q)\big| < \infty. \tag{20}$$

By Dominated Convergence Theorem for Sums, (19) and (20) and pointwise convergence of $C_{fu}^{m}(a,b)$ to $C_{fu}(a,b)$ implies that, as $m\to\infty$, the seventh term of (a.4.1)

$$\sum_{k=0}^{\infty} 2^{k-1}\Big[\gamma_u^m(k-1/2q,k-1/2q) - 2\gamma_u^m(k-1/2q,k+1/2q) + \gamma_u^m(k+1/2q,k+1/2q)\Big]^2$$

$$\to\sum_{k=0}^{\infty} 2^{k-1}\Big[\gamma_u^m(k-1/2q,k-1/2q) - 2\gamma_u^m(k-1/2q,k+1/2q) + \gamma_u^m(k+1/2q,k+1/2q)\Big]^2 < \infty. \tag{21}$$

Similarly, other terms of (a.4.1) converge to the corresponding terms in σ_{22}. Then by (a.4.1), (18) and (21), $\sigma_{22}^m = \sigma_{22}^{m*} \to \sigma_{22} < \infty$ as as $m \to \infty$. Recall that $\sigma_{11}^m = \sigma_{11}^{m*} \to \sigma_{11} < \infty$ and $\sigma_{12}^m = \sigma_{12}^{m*} \to \sigma_{12} < \infty$ as $m \to \infty$ can be proved in similar fashion. This completes the proof. □

A.5 Proof of Theorem 3.6.1

Proof. This proof shows that $\widehat{\sigma}_{22} \xrightarrow{p} \sigma_{22}$ as $l_{sn} \to \infty$. The consistency of $\widehat{\sigma}_{11}$ and $\widehat{\sigma}_{12}$ can be proved in similar fashion. By Theorem 2.2.2,

$$\begin{aligned}\sigma_{22}^m =& \frac{1}{2}\left[2\gamma_1(1/q)\right]^2 + \frac{1}{2}\left[2\gamma_1(1/q) - \gamma_u(1/2q, 1/2q)\right]^2 + \mathrm{I}\{q=2\} \cdot \mathrm{I}\Big\{\lfloor m + 1/q \rfloor \geq 1\Big\} \cdot \left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\ &+ \mathrm{I}\{q \geq 3\} \cdot \mathrm{I}\Big\{\lfloor m + 1/q \rfloor \geq 1\Big\} \cdot \left[\gamma_c(1-1/q) - 2\gamma_c(1) + \gamma_c(1+1/q)\right]^2 \\ &+ \sum_{k=2}^{\lfloor m+1/q \rfloor} \left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2 \\ &+ \sum_{k=1}^{\lfloor m+1/q \rfloor} \left[\gamma_c(k) - \gamma_c(k+1/q) - \gamma_u(1/2q, k-1/2q) + \gamma_u(1/2q, k+1/2q)\right]^2 \\ &+ \sum_{k=1}^{\lfloor m+1/q \rfloor} 2^{k-1}\left[\gamma_u(k-1/2q, k-1/2q) - 2\gamma_u(k-1/2q, k+1/2q) + \gamma_u(k+1/2q, k+1/2q)\right]^2 \\ &+ \sum_{k=2}^{\lfloor m+1/q \rfloor} \sum_{l=1}^{k-1} 2^l \big[\gamma_u(l-1/2q, k-1/2q) - \gamma_u(l+1/2q, k-1/2q) \\ &\qquad - \gamma_u(l-1/2q, k+1/2q) + \gamma_u(l+1/2q, k+1/2q)\big]^2.\end{aligned}$$

By Theorem 2.4.1 and Theorem 3.2.1, for any $m \in \{x/q : x \in \mathbb{N}_+\}$,

$$\widehat{\sigma}_{22}^m \xrightarrow{p} \sigma_{22}^m = \sigma_{22}^{m*} \text{ as } l_{sn} \to \infty. \tag{22}$$

By Theorem 3.4.1,

$$\sigma_{22}^m \to \sigma_{22} \text{ as } m \to \infty. \tag{23}$$

Since $\widehat{\gamma}_c^m(h) = \widehat{\gamma}_c(h)$ and $\widehat{\gamma}_u^m(a,b) = \widehat{\gamma}_u(a,b)$ on the same stream network. For any fixed l_{sn}, the difference of $\widehat{\sigma}_{22}$ and $\widehat{\sigma}_{22}^{m*}$

$$
\begin{aligned}
\widehat{\sigma}_{22}-\widehat{\sigma}_{22}^{m} =& \mathrm{I}\{q=2\}\cdot \mathrm{I}\Big\{\lfloor m+1/q\rfloor<1\Big\}\cdot[\widehat{\gamma}_0(1/q)-2\widehat{\gamma}_c(2/q)+\widehat{\gamma}_c(3/q)]^2\\
&+\mathrm{I}\{q\geq 3\}\cdot \mathrm{I}\Big\{\lfloor m+1/q\rfloor<1\Big\}\cdot[\widehat{\gamma}_c(1-1/q)-2\widehat{\gamma}_c(1)+\widehat{\gamma}_c(1+1/q)]^2\\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)]^2\\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}[\widehat{\gamma}_c(k)-\widehat{\gamma}_c(k+1/q)-\widehat{\gamma}_u(1/2q,k-1/2q)+\widehat{\gamma}_u(1/2q,k+1/2q)]^2\\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}2^{k-1}[\widehat{\gamma}_u(k-1/2q,k-1/2q)-2\widehat{\gamma}_u(k-1/2q,k+1/2q)+\widehat{\gamma}_u(k+1/2q,k+1/2q)]^2\\
&+\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\sum_{l=1}^{k-1}2^{l}\big[\widehat{\gamma}_u(l-1/2q,k-1/2q)-\widehat{\gamma}_u(l+1/2q,k-1/2q)\\
&\qquad -\widehat{\gamma}_u(l-1/2q,k+1/2q)+\widehat{\gamma}_u(l+1/2q,k+1/2q)\big]^2. \qquad \text{(a.6.1)}
\end{aligned}
$$

The next step is to find the limit of $\mathrm{E}(\widehat{\sigma}_{22}-\widehat{\sigma}_{22}^{m})$ as $l_{sn}\to\infty$, which is merely the sum of the limit of the expectations of terms in (a.6.1). Since

$$
\mathrm{E}\left\{\left|\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)]^2\right|\right\}\leq \mathrm{E}\left(\widehat{\sigma}_{22}\right)\leq \sup_{l_{sn}}\mathrm{E}\left(\widehat{\sigma}_{22}\right)<\infty,
$$

by Fubini's Theorem,

$$
\mathrm{E}\left\{\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)]^2\right\}=\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\mathrm{E}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]^2.
$$

Furthermore, by the fact that $\widehat{\gamma}_c(\cdot)$ is a method of moments estimator of $\gamma_c(\cdot)$,

$$
\begin{aligned}
&\mathrm{E}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]^2\\
=&\mathrm{Var}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]+\left\{\mathrm{E}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]\right\}^2\\
=&\mathrm{Var}\left[\widehat{\gamma}_c(k-1/q)-2\widehat{\gamma}_c(k)+\widehat{\gamma}_c(k+1/q)\right]+\left[\gamma_c(k-1/q)-2\gamma_c(k)+\gamma_c(k+1/q)\right]^2. \qquad \text{(a.6.2)}
\end{aligned}
$$

By Cauchy-Schwartz Inequality and the fact, implied by the proof of Theorem 2.4.1, that $\mathrm{Var}[\widehat{\gamma}(\cdot)]\to 0$

as $l_{sn} \to \infty$,

$$\begin{aligned}&\text{Var}\left[\widehat{\gamma}_c(k-1/q) - 2\widehat{\gamma}_c(k) + \widehat{\gamma}_c(k+1/q)\right] \\ \leq& 3\left\{\text{Var}\left[\widehat{\gamma}_c(k-1/q)\right] + 4\text{Var}\left[\widehat{\gamma}_c(k)\right] + \text{Var}\left[\widehat{\gamma}_c(k+1/q)\right]\right\} \to 0\end{aligned}$$

as $l_{sn} \to \infty$. Then, (a.6.2)

$$\text{E}\left[\widehat{\gamma}_c(k-1/q) - 2\widehat{\gamma}_c(k) + \widehat{\gamma}_c(k+1/q)\right]^2 \to \left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2 \text{ as } l_{sn} \to \infty.$$

Then, the first term of (a.6.1)

$$\begin{aligned}\text{E}\left\{\sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\widehat{\gamma}_c(k-1/q) - 2\widehat{\gamma}_c(k) + \widehat{\gamma}_c(k+1/q)\right]^2\right\} =& \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\text{E}\left[\widehat{\gamma}_c(k-1/q) - 2\widehat{\gamma}_c(k) + \widehat{\gamma}_c(k+1/q)\right]^2 \\ \to& \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2,\end{aligned}$$

as $l_{sn} \to \infty$. Similar statement can be made for other terms of (a.6.1), then

$$\begin{aligned}\lim_{l_{sn}\to\infty}\text{E}\left(\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^m\right) \to& \text{I}\{q=2\}\cdot\text{I}\Big\{\lfloor m+1/q\rfloor < 1\Big\}\cdot\left[\gamma_0(1/q) - 2\gamma_c(2/q) + \gamma_c(3/q)\right]^2 \\ &+ \text{I}\{q\geq 3\}\cdot\text{I}\Big\{\lfloor m+1/q\rfloor < 1\Big\}\cdot\left[\gamma_c(1-1/q) - 2\gamma_c(1) + \gamma_c(1+1/q)\right]^2 \\ &+ \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\gamma_c(k-1/q) - 2\gamma_c(k) + \gamma_c(k+1/q)\right]^2 \\ &+ \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\left[\gamma_c(k) - \gamma_c(k+1/q) - \gamma_u(1/2q, k-1/2q) + \gamma_u(1/2q, k+1/2q)\right]^2 \\ &+ \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty} 2^{k-1}\left[\gamma_u(k-1/2q, k-1/2q) - 2\gamma_u(k-1/2q, k+1/2q) + \gamma_u(k+1/2q, k+1/2q)\right]^2 \\ &+ \sum_{k=\lfloor m+1/q\rfloor+1}^{\infty}\sum_{l=1}^{k-1} 2^l\big[\gamma_u(l-1/2q, k-1/2q) - \gamma_u(l+1/2q, k-1/2q) \\ &\qquad - \gamma_u(l-1/2q, k+1/2q) + \gamma_u(l+1/2q, k+1/2q)\big]^2,\end{aligned}$$

which implies that

$$\lim_{m\to\infty}\lim_{l_{sn}\to\infty}\text{E}\left(\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^m\right) = 0. \tag{24}$$

Then for fixed $\epsilon > 0$,

$$\begin{aligned}
\mathrm{P}\left(\left|\widehat{\sigma}_{22} - \sigma_{22}\right| > \epsilon\right) &= \mathrm{P}\left(\left|\,(\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^{m}) + (\widehat{\sigma}_{22}^{m} - \sigma_{22}^{m}) + (\sigma_{22}^{m} - \sigma_{22})\,\right| > \epsilon\right) \\
&\leq \mathrm{P}\left(\left\{\left|\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^{m}\right| > \epsilon/3\right\} \cup \left\{\left|\widehat{\sigma}_{22}^{m} - \sigma_{22}^{m}\right| > \epsilon/3\right\} \cup \left\{\left|\sigma_{22}^{m} - \sigma_{22}\right| > \epsilon/3\right\}\right) \\
&\leq \mathrm{P}\left(\left|\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^{m}\right| > \epsilon/3\right) + \mathrm{P}\left(\left|\widehat{\sigma}_{22}^{m} - \sigma_{22}^{m}\right| > \epsilon/3\right) + \mathrm{P}\left(\left|\sigma_{22}^{m} - \sigma_{22}\right| > \epsilon/3\right)
\end{aligned}$$

By (22), (23), (24) and Markov's inequality,

$$\begin{aligned}
&\lim_{l_{sn}\to\infty} \mathrm{P}\left(\left|\widehat{\sigma}_{22} - \sigma_{22}\right| > \epsilon\right) \\
&= \lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\left|\widehat{\sigma}_{22} - \sigma_{22}\right| > \epsilon\right) \\
&\leq \lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\left|\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^{m}\right| > \epsilon/3\right) + \lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\left|\widehat{\sigma}_{22}^{m} - \sigma_{22}^{m}\right| > \epsilon/3\right) \\
&\quad + \lim_{m\to\infty} \lim_{l_{sn}\to\infty} \mathrm{P}\left(\left|\sigma_{22}^{m} - \sigma_{22}\right| > \epsilon/3\right) \\
&\leq \lim_{m\to\infty} \lim_{l_{sn}\to\infty} \frac{3}{\epsilon}\mathrm{E}\left(\left|\widehat{\sigma}_{22} - \widehat{\sigma}_{22}^{m}\right|\right) + \lim_{m\to\infty} 0 + \lim_{m\to\infty} \mathrm{P}\left(\left|\sigma_{22}^{m} - \sigma_{22}\right| > \epsilon/3\right) = 0.
\end{aligned}$$

Hence $\widehat{\sigma}_{22} \xrightarrow{p} \sigma_{22}$ as $l_{sn} \to \infty$. Recall that the consistency of $\widehat{\sigma}_{11}$ and $\widehat{\sigma}_{12}$ can be proved in similar fashion. This completes the proof. □

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